223 Assignment 7 Solutions

§12.1 #1 Consider $(\ln(y)^z, (\frac{xz}{y})\ln(y)^{z-1}, x\ln(\ln(y))(\ln y)^z)$. Find a function u(x, y, z) so that it has gradient equal to the vector field.

Integrate the first component with respect to x, it is easy because there is no x, so it is a constant. So, we have the first attempt is $x \ln(y)^z + C$. But, once we have this attempt, the rest works out nicely, all from the easy step. We can integrate the second if we use a substitution, say, $u = \ln y$, and we can integrate the third as long as we remember that $a^x = e^{(\ln a)x}$, so $\int a^x dx = \int e^{(\ln a)x} dx = \frac{1}{\ln a} e^{(\ln a)x} + C = \frac{1}{\ln a} a^x + C$. We can apply this to $\int (\ln y)^z dz$ to get the same result. The question asks for one function. I select $x \ln(y)^z + e^{\sqrt{\pi}}$.

§12.1 #2 Show that F(x,y) = (y,2x) is not the gradient vector field of any continuously differentiable function.

Suppose $F(x,y) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$, if this were true then we would have $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ because $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. But in this case, $\frac{\partial F_1}{\partial y} = 1$ and $\frac{\partial F_2}{\partial x} = 2$; $1 \neq 2$.

§12.2 #1 Let f(x, y) = (x, 0).

a. Draw the vector field f in the rectangle $[-2, 2] \times [-2, 2]$.

Tell me if you have challenges with this part.

b. From the picture alone, what can you say of the sign of the integral along the curve $\mathbf{X}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$? Do not compute. Do explain.

Since the vector field is the same for any y value, all that matters is whether the path is going to the left or the right, but the circle goes to the left as much as it goes to the right, furthermore there is as much positive vector field on the right as there is negative on the left, so it cancels for that reason. In either case we get zero.

c. From the picture alone, what can you say of the sign of the integral along the curve $\mathbf{X}(t) = (1 + t)$ $\cos t, \sin t$) $0 \le t \le 2\pi$? Do not compute. Do explain.

This is very similar to the first question, we still get zero, for the first reason, but not for the second reason this time.

§12.2 #2 Let f(x,y) = (x - y, x + y). Plot this vector field by hand or with maple on the rectangle $[-4,4] \times [-4,4]$. Consider the curve $\mathbf{X}(t) = (2\sin t, 2\cos t), \ 0 \le t \le \frac{\pi}{2}$ at the point $(\sqrt{2},\sqrt{2})$. Is the scalar component of the vector field in the direction tangent to the curve at the point positive, negative, or zero? Explain and illustrate your reasoning.

At the point $(\sqrt{2}, \sqrt{2})$ the vector field points to $(0, 2\sqrt{2})$. Be careful - notice the sin and cos are switched. So, $\mathbf{X}(t) = (\sqrt{2}, \sqrt{2})$ at $t = \frac{\pi}{4}$ (which is not surprising), but $\mathbf{X}'(t) = (2\cos t, -2\sin t)$, so $\mathbf{X}'(\pi/4) =$ $(\sqrt{2}, -\sqrt{2})$. Notice the dot product with the vector field is -4 < 0, so the scalar component is negative. This should be pretty visual as well, to see that the curve is going down, while the vector field points up.

§12.3 #1 Evaluate the line integral along the curve
$$\mathbf{X}(t) = (t^3, 3), 1 \le t \le 4$$
 of $f(x, y) = (3, e^{-x^2})$.
 $\int_{\gamma} \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} = \int_{1}^{4} (3, e^{-t^6}) \cdot (3t^2, 0) dt = \int_{1}^{4} 9t^2 dt = [3t^3]_{1}^{4} = 3(64) - 3(1) = 189$

§12.3 #2 Let f(x,y) = (y,x). Evaluate the line integral of this field along the following paths:

a. the line segment from (0,0) to (2,4) parametrised by $\mathbf{X}(t) = (t, 2t), 0 \leq t \leq 2$. $\int_{\mathcal{A}} \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} =$ $\int_0^2 (2t,t) \cdot (1,2) dt = \int_0^2 4t dt = [2t^2]_0^2 = 8.$ b. the line segment from (0,0) to (2,4) parametrised by $\mathbf{X}(t) = (t^2, 2t^2), 0 \le t \le 2. \quad \int_{\gamma} \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} = (t^2, 2t^2) = 0.$

 $\int_{0}^{\sqrt{2}} (2t^2, t^2) \cdot (2t, 4t) dt = \int_{0}^{\sqrt{2}} 8t^3 dt = [2t^4]_{0}^{\sqrt{2}} = 8.$ c. the curve $y = x^2$ from x = 0 to x = 2 parametrised by $\mathbf{X}(t) = (t, t^2) \int_{\gamma} \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} = \int_{0}^{2} (t^2, t) \cdot (1, 2t) dt = 0$

 $\int_0^2 3t^2 dt = [t^3]_0^2 = 8.$

d. the curve $y = x^2$ from x = 0 to x = 2 parametrised by $\mathbf{X}(t) = (t^2, t^4) \int_{\gamma} \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} = \int_0^{\sqrt{2}} (t^4, t^2) \cdot d\mathbf{X}$ $(2t, 4t^3)dt = \int_0^{\sqrt{2}} 6t^5 dt = [t^6]_0^{\sqrt{2}} = 8.$

The point here in (a) and (b) together and (c) and (d) together is independence of parametrisation. The point between the first half and the second half is independence of path. Notice this field is the gradient of g(x, y) = xy.

12.4 #1 Suppose f is a conservative vector field. Is the following statement always true, never true, or sometimes true? Explain your answer.

Suppose that the line integral along the line segment from (-2,0) to (2,0) is 11. If we integrate along the path $(-t, 4 - t^2), -2 \le t \le 2$ then we get -11.

This is always true. The two paths have the same endpoints but in the opposite order. Because the vector field is conservative, the integration of the vector field along a path is only determined based on its endpoints. So, the value should be the same, but with different sign because the order is switched.

12.4 # 2 Let $f = (xy + y\cos(xy), xy = x\cos(xy))$. Explain why the line integral around any closed curve of f is the same as the line integral of (xy, xy).

If we start naively we have that $\oint_{\gamma} \mathbf{f} \cdot d\mathbf{X} = \oint_{\gamma} (xy + y\cos(xy))dx + (xy + x\cos(xy))dy$, but the problem suggests this is equal to $\oint_{\gamma} xydx + xydy$. What happens to the rest? The rest is $\oint_{\gamma} y\cos(xy)dx + x\cos(xy)dy$. Notice this is $\mathbf{g}(x, y) = (y\cos(xy), x\cos(xy))$, which we can also notice is the gradient of $\sin(xy)$. Because it is a gradient, that part of the closed integral is zero. The only remaining part is the rest.