The calculation is similar for the second parametrization:

$$x = \cos(t^2); \qquad dx = -2t\sin(t^2)\,dt; \qquad y = \sin(t^2); \qquad dy = 2t\cos(t^2)\,dt.$$

Check the last (easy) step.

This gives

$$\int_\gamma -y\,dx + x\,dy = \int_0^{\sqrt{\pi}} \left(\sin^2(t^2) + \cos^2(t^2)\right) 2t\,dt = \pi.$$

The results are the same, as they should be.

The general case A general argument for independence of parametrization in line integrals resembles the specific calculation in Example 3. Suppose we are given any two differentiable parametrizations

$$\mathbf{X}_1(s) = (x_1(s), y_1(s)) \quad \text{and} \quad \mathbf{X}_2(t) = (x_2(t), y_2(t))$$

for the curve γ, where $a \le s \le b$ and $c \le t \le d$. For the line integral $\int_\gamma P\,dx + Q\,dy$, these parametrizations lead, respectively, to two ugly integrals:

$$I_1: \quad \int_a^b \left(P(\mathbf{X}_1(s))\,\frac{dx_1}{ds} + Q(\mathbf{X}_1(s))\,\frac{dy_1}{ds} \right) ds$$

$$I_2: \quad \int_c^d \left(P(\mathbf{X}_2(t))\,\frac{dx_2}{dt} + Q(\mathbf{X}_2(t))\,\frac{dy_2}{dt} \right) dt.$$

These integrals look complicated, to be sure, and much notation is involved. But here we care only that I_1 and I_2 have the *same* value. To see this one shows first that s is a differentiable function of t such that $\mathbf{X}_2(t) = \mathbf{X}_1 \circ s$. In other words,

$$\mathbf{X}_2(t) = \left(x_2(t), y_2(t)\right) = \mathbf{X}_1(s(t)) = \left(x_1(s(t)), y_1(s(t))\right).$$

(Showing in general that $s(t)$ is differentiable is slightly delicate. In Example 3, we had simply $s = t^2$, which is clearly differentiable.) In particular, $s(c) = a$ and $s(d) = b$.

Assuming the foregoing facts, we can make the change of variable $s = s(t)$ in I_1, which (despite its complicated appearance) is an ordinary integral of the single-variable type. The happy result is that I_2 emerges:

$$\begin{aligned} I_1 &= \int_a^b \left(P(\mathbf{X}_1(s))\,\frac{dx_1}{ds} + Q(\mathbf{X}_1(s))\,\frac{dy_1}{ds} \right) ds \\ &= \int_c^d \left(P\left(\mathbf{X}_1(s(t))\right)\frac{dx_1}{ds} + Q\left(\mathbf{X}_1(s(t))\right)\frac{dy_1}{ds} \right) \frac{ds}{dt}\,dt \\ &= \int_c^d \left(P(\mathbf{X}_2(t))\,\frac{dx_2}{dt} + Q(\mathbf{X}_2(t))\,\frac{dy_2}{dt} \right) dt = I_2. \end{aligned}$$

(We used the ordinary chain rule in the last line, in the form

$$\frac{dx_1}{ds} \cdot \frac{ds}{dt} = \frac{d}{dt}\left(x_1(s(t))\right) = \frac{dx_2(t)}{dt}$$

for the first summand and similarly in y for the second.) We conclude that $I_1 = I_2$, just as desired.

The fundamental theorem for line integrals

One version of the fundamental theorem of elementary calculus relates the integral and the derivative: If a function f and its derivative f' are continuous on $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

(There are other ways to state the fundamental theorem of calculus; this version is best for present purposes.)

A similar "fundamental theorem" holds for line integrals. To ensure that all the ingredients exist, we assume that the function $h(x, y)$ has continuous partial derivatives on and near the curve γ, and that γ is smooth, except perhaps at its endpoints.

THEOREM 2 (Fundamental theorem for line integrals) Let $h(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let γ be a smooth, oriented curve starting at $\mathbf{X}_0 = (x_0, y_0)$ and ending at $\mathbf{X}_1 = (x_1, y_1)$. Then

$$\int_\gamma \nabla h \cdot d\mathbf{X} = h(\mathbf{X}_1) - h(\mathbf{X}_0).$$

If γ is a closed curve (i.e., $\mathbf{X}_1 = \mathbf{X}_0$), then

$$\int_\gamma \nabla h \cdot d\mathbf{X} = 0.$$

Proof The proof is a pleasing and straightforward exercise with the chain rule. Let the curve γ be parametrized, as usual, by a function

$$\mathbf{X}(t) = \big(x(t), y(t)\big); \qquad a \le t \le b.$$

(Because γ is smooth, we can assume that x' and y' are continuous functions, and thus all the needed integrals exist.) Then the line integral has the form

$$I = \int_\gamma \nabla h \cdot d\mathbf{X} = \int_a^b \Big(h_x(\mathbf{X}(t)), h_y(\mathbf{X}(t))\Big) \cdot \mathbf{X}'(t)\,dt.$$

Now the composite $h(\mathbf{X}(t))$ is a new function of t, and, by the chain rule in several variables,

$$\frac{d}{dt}\Big(h(\mathbf{X}(t))\Big) = \nabla h(\mathbf{X}(t)) \cdot \mathbf{X}'(t).$$

In other words, the integrand in the last integral above is the t-derivative of $h(\mathbf{X}(t))$. Therefore, by the ordinary fundamental theorem of calculus,

$$I = \int_a^b \frac{d}{dt}\Big(h(\mathbf{X}(t))\Big)\,dt = h(\mathbf{X}(b)) - h(\mathbf{X}(a)) = h(x_1, y_1) - h(x_0, y_0),$$

as claimed. ■