Mathematics 228 PS1 Solutions

1.2 66.

$M(t) = P(t)W(t) = (2 \times 10^6 + 2 \times 10^4 t)(80 - 0.005t^2)$
$= -100t^3 - 10^4 t^2 + 1.6 \times 10^6 t + 1.6 \times 10^8$

Graph P(t) on $[0..100]\times[0..5\times10^6]$ to find the population growing linearly from 2 to 4 million in 100 years.
Graph W(t) on $[0..100]\times[0..90]$ to find the weight decreasing from 80 kg to 30 kg, with the decrease becoming more rapid as the years go by.
Graph M(t) on $[0..100]\times[0..3\times10^8]$ to find the total mass increasing from 1.6×10^8 for the first 47 years (about) to a little above 2×10^8 kg then falling more rapidly to 1.2×10^8 after 100 years.

1.5 60. a. The ratio of the next to the original is $\frac{404}{400} = 1.01$. b. The second notable tone would then be 1.01(404) = 408.04 Hz. c. We have a DTDS: $T_{n+1} = 1.1T_n$. The values are in this table

n	T_n
0	400
1	404.00
2	408.04
3	412.12
4	416.24
5	420.40

So, the fifth perceivable tone is 420 Hz

d. For the musician the ratio is $\frac{400.5}{400} = 1.00125$, the DTDS is
$T_{n+1} = 1.00125T_n$ and the fifth value is 402.5, still imperceptible to the non-musician.

1.6 46. $b_{t+1} = rb_t - 10^6$, so $b^* = rb^* - 10^6$ yields $b^* = \frac{10^6}{r - 1}$, a positive value if $r > 1$.

1.7 54. The equation is $P(t) = 1600(2^{-t/43})$, so $P(t+1) = 2^{-1/43}P(t)$.

1.7.58. Our goal is to plot t v. ln(P(t)), so we plot t and $\ln(1600(2^{-t/43}))$, which equals $\ln(1600) - \frac{t}{43}\ln(2)$ on the window $[0..100]\times[0..10]$. The semilog graph is linear. It has gone down to half when we reach ln(800) which is about 6.68. This happens at 43 years later as stated. Note how the height is not half the height - this is the log, but it is ln(2) less [ln(800 = ln(1600/2) = ln(1600) - ln(2)].

1.9..34. DTDS: $c_{t+1} = 0.8(c_t + S) + 0.2(0.0004)$

solve to find $c^* = 4S + 0.0004$. So, if $c^* = 0.04$, then $S = 0.0099$

1.9.46. DTDS: $s_{t+1} = \left(\frac{3.3}{3.45}\right)s_t + \left(\frac{0.3}{3.45}\right)(0.001)$

solve to find $s^* = 0.002$, twice the inflowing concentration because of the evaporation raising the salinity.

1.10.42 DTDS: $M_{t+1} = M_t - \left(\frac{0.5}{1.0 + \alpha M_t}\right)M_t + 1.0$

solve to find $M^* = \frac{1}{.5 - \alpha}$ Graph on $[0..0.4]\times[0..10]$. When $\alpha > 0.5$ the values become negative and meaningless. Note as α approaches 0.5 we have a higher and higher concentration (less is being used, so more remains). At 0.5 and higher there is no actual equilibrium, the concentration increases without bound. Cobwebbing reinforcs these results, leading to the equilibrium of 2.5 for $\alpha = 0.1$ and no equilibrium for $\alpha = 1$.

1.10.45. Graph r(b) on $[0..5\times10^6]\times[0..3]$

DTDS: $b_{t+1} = 2e^{-\frac{b_t}{10^6}}b_t$; graph on $[0..5\times10^6]\times[0..5\times10^6]$

Unsurprisingly one equilibrium is at zero. Another is at $b^* = 10^6\ln 2$ or about 693147. The zero one is not stable, but the nearly 700,000 one is.

1.11.18. Suppose we're considering:

$$N_{r+1} = \begin{cases} 1.65N_t - 1000 & \text{if } N_t > 1000 \\ 1.65N_t & \text{if } N_t \le 1000 \end{cases}$$

To find the equilibrium we try

$N^* = 1.65N^* - 1000$ to get 1538

Looking at the updating function graph, examine what happens when N_0 is close to 1000, but just below. It jumps past the equilibrium point. To see with values: Suppose $N_0 = 950$. Then $N_1 = 1567.5$, $N_2 = 1586.4$, $N_3 = 1617.5$, and we see the values continue to increase, with the updating function now above the diagonal (once we get above the diagonal, there's no stopping it from growing – and we got above the diagonal when we jumped to 1567 for N_1.

2.8.44. $\frac{db}{dt} = e^{-b(t)}$ says that the rate of increase of the population is decaying exponentially with the increase of the population. In other words, as the population grows the rate of increase of the population drastically gets closer to 1. The solution $b(t) = \ln(t)$ is an increasing function To check that it works, we compute:

$$\frac{d\ln(t)}{dt} = \frac{1}{t}$$

$e^{-\ln(t)} = \frac{1}{e^{\ln(t)}} = \frac{1}{t}$, they are, indeed, equal, so $\ln(t)$ is in fact a solution.

3.1.42 Graph $V_{t+1} = \begin{cases} 0.8V_t & \text{if } 0.8V_t > 20 \text{ if } V_t > 25 \\ 0.8V_t + 10 & \text{if } 0.8V_t \le 20 \text{ if } V_t \le 25 \end{cases}$ along with the diagonal. Notice the diagonal passes through the gap in the graph. This says there is no equilibrium. We may check by attempting to find one algebraically: $V^* = 0.8V^* + 10$ yields 50, but that equation isn't used for $V_t = 50$. In fact, $V^* = 0.8^2V^* + 10$ yields 36, where the equation still isn't used. $V^* = 0.8^3V^* + 10$ yields 20.49 which produces our 3:1 equilibrium situation.

3.2.32. Correction to book: suppose the temperature produced is a linear function of the temperature on the thermo**stat**.

a. Call T the temperature produced and z the theromostat setting. Then $T = 2.5z - 29$ from the two given points (20, 21) and (19, 18.5).

b. We respond to 18.5° by setting the thermostat to 21.5°. This produces a temperature of 24.75°. We're overreacting on both sides, so we now say *curse this*, I'm going lower to fix things quickly. So, we set it to 15.25°. This produces a temperature of 9.125°. This is cold indoors. So we want to heat things up quickly and we ask for 30.875°. We get more than we bargain for with 48.1875°. If you're not used to C, I recommend converting these numbers to F to see how extreme they are.

Day	Setting before	effect
0	20	21
1	19	18.5
2	21.5	24.74
3	15.25	9.125
4	30.875	48.1875

c. $z_t = 20 + (20 - T_t) = 40 - T_t$

d. $T_{t+1} = 2.5z_t - 29 = 2.5(40 - T_t) - 29 = 71 - 2.5T_t$

Because the slope is less than –1, the system is unstable – no surprise there. If only we had been smart enough to set the thermostat at 19.6.

3.2.38

year	seeds	size	seeds/adult
0	20	4	3
1	60	1.5384	.5384
2	32.304	2.6807	1.6807
3	54.293	1.6866	.6866

$n_{t+1} = n_t\left(\frac{100}{n_t + 5} - 1\right)$. We solve for an equilibrium to find $n^* = 45$.

To check we take a derivative and evaluate at $n_t = 45$

$$\frac{dn_{t+1}}{dn_t} = \left(\frac{100}{n_t + 5} - 1\right) + n_t\left(-\frac{100}{(n_t + 5)^2}\right)$$

at $n_t = 45$ gives $\frac{dn_{t+1}}{dn_t} = -\frac{4}{5}$

This is just barely above -1 (hard to see on a graph), but supports our data above which appear to be slowly converging to 45.