233 Problem Set 1 Solutions

1. Describe the intersection of the three planes $u+v+w+z=6$ and $u+w+z=4$ and $u+w=2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u=-1$ is included? Find a fourth equation that leaves us with no solution.

This is intended to be before we started matrices, so I will try to do without them. Using the second plane, the first tells us $v+4=6$, so $v=2$. Using the third plane, the second tells us $2+z=4$, so $z=2$, also. Using only the third plane, we know that $w=2-u$. So, u can be anything, but all other variables are determined. This gives us one dimension of freedom, so the solution is a line. If we also include the fourth plane, then we know that $w=3$ and hence the intersection is only one point. A simple fourth equation would be $v=0$ which cannot coexist with $v=2$, or maybe $u+w=4$ is easier to see since it only requires comparing to the original list.

2. Under what conditions on y_0, y_1, y_2 do the points $(0,y_0),(1,y_1),(2,y_2)$ lie on a straight line?

Notice that the three points have different x-coordinates. So, they are not a vertical line. This is good news, because therefore we may assume they are on a line given by $y=mx+b$ and then see the consequences. The first point tells us $y_0=b$, which looks pretty useful. From that the second two points say $y_1=m+y_0$ and $y_2=2m+y_0$. The first gives $m=y_1-y_0$ and the second gives $m=\frac{1}{2}(y_2-y_0)$. For these to all be on one line these m values need to be equal, so we have $y_1-y_0=\frac{1}{2}(y_2-y_0)$ or perhaps nicer, $2y_1=y_2+y_0$, which seems to make sense, because it tells us that the middle one is the average of the outside two.

3. Below is a system of three linear equations in three unknowns with two symbolic constants, a and b. By changing the values of a and b, we can get not only different solutions but also different numbers of solutions. Use your calculator to reduce the augmented matrix of this system in order to answer the questions. $x+2y+az=4$, $2x-y+3z=b$, $3x-4y+2z=-3$

I carefully used my calculator, performing each operation one at a time. I almost went to row echelon form, but saved the last division step. This produces the matrix:

$$\begin{bmatrix} 1 & 2 & a & 4 \\ 0 & 1 & \frac{2a-3}{5} & \frac{8-b}{5} \\ 0 & 0 & a-4 & 1-2b \end{bmatrix}$$

For what values of a and b have (a) no solutions? (b) a unique solution? (c) a line of solutions (i.e. one free variable)? (d) a plane of solutions (i.e. two free variables)?

This has a unique solution if we can divide by $a-4$, which means as long as $a\neq 4$ there is a unique solution. Because then we can reduce to echelon form with three leading ones. If that isn't the case, and $a=4$ then we have the final equation saying $0=1-2b$. This will be inconsistent unless $b=\frac{1}{2}$. So, there are no solutions when $a=4$ and $b\neq\frac{1}{2}$. If the entire last row is $0=0$ (which happens when $a=$ and $b=\frac{1}{2}$), then there is one free variable (the last one) and we have a line of solutions. Those are all possible cases, since there are leading ones in both the first and second variables, there cannot be a plane of solutions.

4. Which of these rules gives a correct definition of the rank of A?

(a) The number of nonzero rows in the row reduced matrix R.

This works. It does give us a good definition, because any nonzero row will have a leading one in reduced form, and we said a way to find rank was to count leading ones, thus giving the number of non-free variables.

(b) The number of columns minus the total number of rows in A.

No, this does not work. Here is an example to see.

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

The rank of this matrix is 1 because we can easily row reduce it to one nonzero row and then use the answer in part (a). The number of columns minus the number of rows gives $4-2=2\neq 1$

(c) The number of columns minus the number of free columns in A.

This is a good definition. The number of columns is the number of variables, if we subtract the number of free variables, then we get the number of non-free variables, which we said in class was a definition of rank.

(d) The number of 1s in R.

No, this does not work. Here is an example to see

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 1 by the answer in (a) since this is the reduced form. There is nothing more we can do to it. However, there are 4 1s. $4 \neq 1$.

5. Use the information given to determine whether each linear system $Ax = b$ is consistent. If so, state the number of parameters in its general solution: Size of A, Rank(A), Rank$[A|b]$

If Rank(A) is one less than Rank$[A|b]$, then when row reducing $[A|b]$ there is a row in A that has zeroes with a nonzero entry in b. This makes the system $Ax = b$ inconsistent. If not inconsistent, then the number of free variables (parameters) = the number of variables (columns) - the rank. Knowing all of this makes this problem quite simple.

a) 3×3, 3, 3. So, consistent, and $3 - 3 = 0$ parameters, a unique solution.

b) 3×3, 2, 3. Inconsistent.

c) 3×3, 1, 1. Consistent, and $3 - 1 = 2$ parameters, planar solution.

d) 5×9, 2, 2. Consistent, and $9 - 2 = 7$ parameters in the sotuion.

e) 5×9, 2, 3. Inconsistent.

f) 4×4, 0, 0. Consistent, and $4 - 0 = 4$ parameters in the solution.

g) 6×2, 2, 2. Consistent, and $2 - 2 = 0$ parameters, a unique solution.

6. By trial and error find examples of 2×2 matrices such that

(a) $A(A) = -\mathbf{I}$, A having only real entries Notice that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

(b) $B(B) = \mathbf{0}$, although $B \neq \mathbf{0}$

Notice that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

(c) $CD = -DC$, but $CD \neq \mathbf{0}$

Inspired by example 2.24:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \text{ but } \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}.$$

(d) $EF = \mathbf{0}$, although no entries of E or F are zero. Notice

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

7. A is 3×5, B is 5×3, C is 5×1, and D is 3×1. All entries are 1. Which of these matrix operations are allowed, and what are the results? For those not allowed, explain why not.

$$BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & ! \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & ! \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$$

$$ABD = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

DBA is not possible since the number of columns of D (1) is not equal to the number of rows of B (5).
$A(B + C)$ is not possible since the sizes of B and C are different.

8. Let L be a letter L matrix, which is a square matrix of 0s and 1s in which the 1s form the shape of the letter L. For any square matrix A, experiment using your calculator with the product AL for A and L of various sizes, such as 4×4, 5×5, and 6×6, to help you answer the following questions. Your answers must be valid for letter L matrices of every possible size. (a) Describe the product AL, that is, what does L do to A when it is multiplied on the right of A? Explain. (b) What is L^2? (c) Find a general formula for L^p in terms of p and n, where L is $n \times n$.

This answer looks a little different, but shows more of the computation. The solution is on the next two pages. Please ask me if you have trouble reading this format. If so, I will not do it again.

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (4.1)$$

> $A := Randmat(6, 4);$

$$A := \begin{bmatrix} 6 & -9 & -4 & 3 \\ 4 & -3 & -3 & -7 \\ 4 & 5 & 5 & 9 \\ 2 & 9 & -2 & -7 \\ 7 & 7 & -7 & -5 \\ -5 & 0 & 0 & -8 \end{bmatrix} \quad (4.2)$$

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> A.L;
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$$\begin{bmatrix} -4 & 3 & 3 & 3 \\ -9 & -7 & -7 & -7 \\ 23 & 9 & 9 & 9 \\ 2 & -7 & -7 & -7 \\ 2 & -5 & -5 & -5 \\ -13 & -8 & -8 & -8 \end{bmatrix} \quad (4.3)$$

(a) Describe the product $A\,L$. That is, what does L do to A when it is multiplied on the right of A? Explain.

Because of all the ones on the bottom, AL copies the right most column and puts it in each of the columns after the first one. Because of all the ones on the left, the leftmost column is the sum of all the entries in that row.

(b) What is L^2?

> $L := LetterL(6);$

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (4.4)$$

> $L.L;$

(4.5)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.5}$$

Thinking about applying the above rule to L itself, the columns after the first have 1s on the bottom, because they're all copies of the last colum (it's hard to tell, but that's where it came from). And on the left we have the sum of the rows. Because of all the 0s, the rows aside from the last one will all have 1s, and the last one will have n, the size of the L-matrix.

(c) Find a general formula for L^p in terms of p and n, where L is n by n.

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> L.L.L;
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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 11 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.6}$$

Student Workspace

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> L.L.L.L;
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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 16 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.1.1}$$

For all the reasons above, nothing changes as we take powers, except for the bottom left entry. We see above in L^2, that entry is n (just count the ones). In L^3 it is n + (n-1), and in L^4 it is n+2 (n-1). In general for p > 1, we have then that the lower left entry is n+(p-2)(n-1). Out of curiosity, what does this do if p = 1? The correct answer is 1 for that entry. n+(-1)(n-1) = 1, that's nice. Does it work for p = 0? Just like with numbers, anything to the zero power should be the identity, so the correct answer is 0 for the lower left entry. What do we get? n+(-2)(n-1) = n-2n+2 = 2-n. Alas. Apparently that's too much to ask for.

3.5 Maple Problems