First case of completeness

We have a temporary definition of limit. A number L (in some representation, might end with infinitely many 9s), is the limit of the increasing sequence $\{a_n\}$ (here none of the a_n end with infinitely many 9s) if given any integer $k > 0$, there is an N_k such that if $n > N_k$, then a_n agrees with L to k decimal places.

First completeness theorem (1.3): A positive increasing sequence $\{a_n\}$ (here none of the a_n end with infinitely many 9s) which is bounded above by M has a limit.

Mattuck says that his proof isn't "formal". Mine probably isn't either, but, I think it's better. And I think we should be "better" now.

Suppose that $\{a_n\}$ (here none of the a_n end with infinitely many 9s) is positive increasing sequence which is bounded above by M. Consider $\{int(a_n)\}$ (By $int(x)$ I mean the integer part of x.). In this sequence of integers, there are no more than $int(M)$ changes, but there are infinitely many terms. Because $a_n \leq a_{n+1}$ they are in order. Eventually the integer parts must settle. This is a version of the pigeonhole principle. There infinitely many $\{int(a_n)\}$ to be put in the finitely many (actually there are $int(M) + 1$) spots up to $int(M)$. So, because there are only finitely many numbers in $\{int(a_n)\}$ one of them must be the largest. This is the integer part of L.

This is the case $k = 0$. We have proven that for $k = 0$ there is an N_0 (the first time when $\{int(a_n)\}$ is its largest value) such that if $n > N_0$, then a_n agrees with the integer part of L.

This is our base case for induction. We will continue with the induction step. The remaining steps are about the same, maybe a little bit easier.

So, as typical for induction, we assume our result is true for k and attempt to prove it for $k+1$ for any k. Our induction hypothesis is there is L and N_k such that if $n > N_k$, then $trunc_k(a_n)$ (By $trunc_k(x)$ I mean x truncated at the k decimal place) agrees with $trunc_k(L)$. We now want to extend this to $k + 1$. Therefore we look at the next digit. Now consider $trunc_{k+1}(a_n)$. This agrees in the first k places for $n > N_k$. Now focus all attention on the $k + 1$ decimal place. Like before, there are a limited number of options, this time only 10 (0 to 9). Sometime it hits the largest value of all the a_n with $n > N_k$, and because $a_n \leq a_{n+1}$, it doesn't go down from there. Suppose it hits the largest value at N_{k+1}. Use the largest value for the $k+1$ place of L. We then have that there is L and N_{k+1} such that if $n > N_{k+1}$, then $trunc_{k+1}(a_n)$ agrees with $trunc_{k+1}(L)$.

Question: where did we use that none of the a_n end with infinitely many 9s?