Partial Ring of Quotients $Q(R,T)$

On problem 21.12 Fraleigh says to spend 15 minutes making sense of the partial ring of quotients. I want to help with that. He says instead of starting with an integral domain, as we do for the field of fractions, we can start with a nonzero commutative ring. So, there may not be unity, and there may be zero divisors. For example, $R$ could be $4\mathbb{Z}_{100}$ which contains 25 elements

$$R = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96\},$$

and let $T$ by a nonempty subset of $R$ closed under multiplication without 0 or 0 divisors. Our 0 divisors here are 20, 40, 60, and 80. So, we avoid those. Let $T = \{16, 36, 56, 76, 96\}$. Remember multiplying by something ending in 6 yields something that ends in 6, so this is multiplicatively closed.

When working with people in office hours, I made a mistake at the next step. For those that were there, I apologise, and I am trying to make things right by doing this. Instead of forming $D \times D$ or $R \times R$ in Step 1 (p. 191) we will use $R \times T$ (some of you said this was the intent, I am most sorry to you). So, first consider $R \times T$: it has elements like $(4, 16)$ and $(44, 56)$. This is playing the role of $S$. Next consider putting the same equivalence relation on this set as we did with our field of fractions, so

$$[(4, 16)] = \{(4, 16), (64, 56), (24, 96), (84, 36), (44, 76)\}.$$

I found these by multiplying the first and second elements both by 16 (mod 100). $[(4, 16)]$ is one element of $Q(R,T)$ that we can write in five different ways. Because we are avoiding zero divisors in the second component we still have transitivity in our equivalence relation, which we need.

For step 2, we can still define multiplication and addition in the same way. Here are two examples:

$$[(4, 16)] + [(44, 56)] = [(4(56) + 16(44), 16(56))] = [(28, 96)] = [(8, 56)]$$

$$[(4, 16)][(44, 56)] = [(4(44), 16(56))] = [(76, 96)] = [(16, 36)]$$

Notice here why we needed the second elements to be multiplicatively closed, and the last step was not necessary, but I tried to make an equivalent representative from the equivalence class that had smaller numbers. See that whenever we compute between two elements in $Q(R,T)$ we stay in the set, so we do have a ring in this way. It's worth noticing that

$$[(0, 16)] = \{(0, 16), (0, 36), (0, 56), (0, 76), (0, 96)\}$$

is the additive identity.

In what sense is this an "enlargement" of $R$? That's a good question. $4 \in R$ but surely $4 \notin Q(R,T)$. And, if we follow Step 4 (p. 194) even $i(r) = [(r, 1)] \notin Q(R,T)$. This reminds me of something that Matt C asked about on Friday. He asked if there were other maps that could be used for this purpose, and I said "Yes, like $i(r) = [(2r, 2)]$". It's exactly one of these maps that is needed here. So, instead of $i(r) = [(r, 1)]$ for this ring of partial quotients we will use $i(r) = [(16r, 16)]$ which seems reasonably equivalent to $r$, much like $[(r, 1)]$ is reasonably equivalent to $r$.

You are asked to show that there is a unity in this ring, so I won't show you that in this example, although I think you can easily find it. You are also asked to show that nonzero elements **of** $T$ (I almost missed that detail, this is not for all of $R$) are units, and I think that isn't too bad also, as long as you keep in mind what I said above. This doesn't *actually* mean elements of $T$ but their copies in $Q(R, T)$. So, $i(t) = [16t, 16]$. For example, $i(56) = [(96, 16)]$, and I think if you see what the identity is above, you should be able to see what the inverse of $[(96, 16)]$ is.

Is $Q(R, T)$ a field? Well, likely not given that the problem calls it a partial ring of quotients. Why not? Two reasons. First, there are still zero divisors: $[(20, 16)][(40, 96)] = [(0, 36)] = [(0, 16)]$. Second, because there are zero divisors there are elements without inverses. I won't show this because I'd need to tell you what the unity is.

I tried hard when writing this up to *not* spoil any of the problems. I was tempted to complete one for you, but then I felt bad that I was stealing your options. Please ask me if there are any questions. You're all great - sometimes better than I am. Thank you for your hard work.

One more important thing – you don't need to *explain* any of this in doing 21.12. Feel free to use it in that or any of the subsequent problems.