CHAPTER 2.8

Differential Systems

EXAMPLE 1. We are given two tanks of capacity 100 gallons, each filled with a mixture of salt and water. The tanks are connected by pipes as shown in Fig. 2.52 and at all times the mixture in each tank is kept uniform by stirring.

The mixture from tank I flows into tank II through a pipe at 10 gal/min, and in the reverse direction, the mixture flows into tank I from tank II through a second pipe at 5 gal/min. Also, the mixture leaves tank II through a third pipe at 5 gal/min, while fresh water flows into tank I through another pipe at 5 gal/min.

Denote by $x(t)$ the amount of salt (in lbs) in tank I at time t, and by $y(t)$ the corresponding amount in tank II. Suppose, at time $t = 0$, there are x_0 lbs of salt in tank I, and 0 lbs of salt in tank II. Find expressions for $x(t)$ and $y(t)$ in terms of t.

Consider the time interval from time t to time $t + \Delta t$. During that time interval, each gallon flowing into tank I from tank II contains $y(t)/100$ lbs of salt, while each gallon flowing from tank I to tank II contains $x(t)/100$ lbs of salt. Hence, the net change of the amount of salt in tank I during the time interval is

$$\Delta x = \frac{5y(t)\Delta t}{100} - \frac{10x(t)\Delta t}{100},$$

while the corresponding change for tank II is

$$\Delta y = \frac{10x(t)\Delta t}{100} - \frac{10y(t)\Delta t}{100}.$$

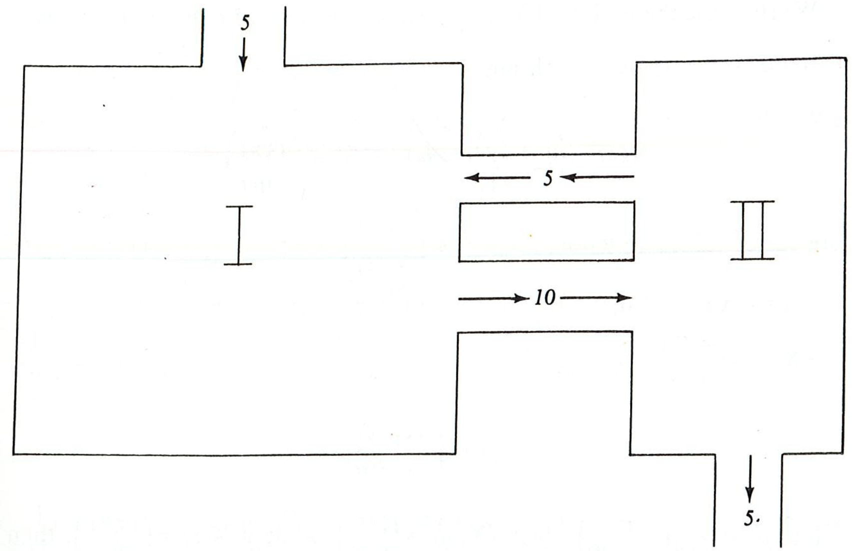

Figure 2.52

Dividing both equations by Δt and letting $\Delta t \to 0$, we get

$$\begin{cases} \dfrac{dx}{dt}(t) = \dfrac{5}{100}\, y(t) - \dfrac{10}{100}\, x(t), \\ \dfrac{dy}{dt}(t) = \dfrac{10}{100}\, x(t) - \dfrac{10}{100}\, y(t). \end{cases} \tag{1}$$

In addition, we know that

$$x(0) = x_0, \qquad y(0) = 0. \tag{2}$$

The functions $t \to x(t)$, $t \to y(t)$ must be determined from conditions (1) and (2).

A system of equations involving two unknown functions x and y which has the form

$$\begin{cases} \dfrac{dx}{dt} = ax + by, \\ \dfrac{dy}{dt} = cx + dy, \end{cases} \tag{3}$$

where a, b, c, d are given constants, is called a *differential system*.

Thus (1) is a differential system with $a = -\frac{10}{100}$, $b = \frac{5}{100}$, $c = \frac{10}{100}$, $d = -\frac{10}{100}$. The condition

$$x(0) = x_0, \qquad y(0) = y_0,$$

where x_0, y_0 are given constants, is called an *initial condition* for the system (3). Thus (2) is an initial condition.

We shall use the notion of a vector-valued function of t. A vector-valued function $\mathbf{X}(t)$ assigns to each number t a vector $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Thus,

$$\mathbf{X}(t) = \begin{pmatrix} t^2 \\ t^3 + 1 \end{pmatrix} \quad \text{and} \quad \mathbf{X}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

are vector-valued functions. If $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, then $t \to x_1(t)$ and $t \to x_2(t)$ are scalar-valued functions. We define the *derivative* of the function $t \to \mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ by

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} dx_1/dt \\ dx_2/dt \end{pmatrix}.$$

Thus, if $\mathbf{X}(t) = \begin{pmatrix} t^2 \\ t^3 + 1 \end{pmatrix}$, then $d\mathbf{X}/dt = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$, while if $\mathbf{X}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, then $d\mathbf{X}/dt = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. Note that $d\mathbf{X}/dt$ is again a vector-valued function.

Exercise 1. Fix a vector $\mathbf{Y}$. Define a vector-valued function $t \to \mathbf{Y}(t)$ by setting $\mathbf{Y}(t) = t^n\mathbf{Y}$. Show that

$$\frac{d\mathbf{Y}}{dt} = nt^{n-1}\mathbf{Y}.$$

Now let the scalar-valued functions $t \to x(t)$, $t \to y(t)$ be a solution of the differential system (3). In vector form, we can write

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}. \tag{4}$$

We define the vector-valued function $t \to \mathbf{X}(t)$ by $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then the left-hand side of (4) is $d\mathbf{X}/dt$, and the right-hand side of (4) is

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\mathbf{X}(t)).$$

Thus (4) may be written in the form

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{X}(t). \tag{5}$$

How shall we solve Eq. (5) for $\mathbf{X}(t)$? Recall that letting a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ act on a vector $\mathbf{X}$ to give the vector $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{X}$ is analogous to multiplying a number x by a scalar a to give the number ax. So Eq. (5) is analogous to the equation

$$\frac{dx}{dt} = ax, \tag{6}$$

where x is now a scalar-valued function of t and a is a given scalar. We know how to solve Eq. (6). The solutions have the form

$$x(t) = Ce^{at},$$

where C is a constant. Setting $t = 0$, we get $x(0) = C$, so

$$x(t) = x(0)e^{at} = e^{ta}(x(0)),$$

where we have changed the order of multiplication with malice aforethought. Let us look for a solution to Eq. (5) by looking for an analogue of $e^{ta}(x(0))$. We take

$$\mathbf{X}(t) = e^{tm}(\mathbf{X}(0)), \quad \text{with} \quad m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7}$$

First we must define the *exponential of a matrix*. In §1 we shall define, given a matrix m, a matrix to be denoted e^m or $\exp(m)$ and to be called the *exponential of* m.

Applying the matrix e^{tm} to a fixed vector $\mathbf{X}(0)$, we then obtain a vector for each t, and thus we get the vector-valued function $t \to \mathbf{X}(t)$ defined in (7). We shall then show that $\mathbf{X}(t)$ solves (5).

In what follows we shall use the symbol I for *the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is properly denoted $m(I)$. This simplifies the formulas, and should cause no confusion.

§1. The Exponential of a Matrix

Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix. Since we have defined addition and multiplication of matrices, we can write expressions such as m^2 or $m^3 - 3m + I$. We interpret $m^3 - 3m + I$ as the result of applying the polynomial $P(x) = x^3 - 3x + 1$ to the matrix m:

$$P(m) = m^3 - 3m + I.$$

More generally, if $Q(x)$ is the polynomial

$$Q(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0,$$

where $c_n, c_{n-1}, \ldots, c_1, c_0$ are scalars, we set

$$Q(m) = c_nm^n + c_{n-1}m^{n-1} + \cdots + c_1m + c_0I,$$

and we regard $Q(m)$ as the matrix obtained by *applying the polynomial Q to the matrix m*.

We now replace the polynomial Q by the *exponential function* $\exp(x)$. We know that $\exp(x)$ is given by an infinite series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \tag{8}$$

where the series converges for every number x. We wish to apply the function $\exp(x)$ to the matrix m. We *define*

$$\exp(m) = I + m + \frac{m^2}{2!} + \cdots + \frac{m^n}{n!} + \cdots. \tag{9}$$

An infinite series is understood as a *limit*. Thus, Eq. (8) means that the sequence of numbers

$$1, 1 + x, 1 + x + \frac{x^2}{2!}, \ldots, 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \ldots$$

converges to the limit $\exp(x)$ as $n \to \infty$. Similarly, we interpret Eq. (9) to say that $\exp(m)$ is defined as the *limit of the sequence of matrices*

$$I, I + m, I + m + \frac{m^2}{2!}, \ldots, I + m + \frac{m^2}{2!} + \cdots + \frac{m^n}{n!}, \ldots.$$

Of course, $\exp(m)$ is then itself a matrix.

EXAMPLE 2. Let m be the diagonal matrix

$$m = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix},$$

where s, t are scalar. What is $\exp(m)$? Recall the formula for $\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}^n$ found in Chapter 2.7.

$$1 + m + \frac{m^2}{2!} + \cdots + \frac{m^n}{n!}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} s^2 & 0 \\ 0 & t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} s^3 & 0 \\ 0 & t^3 \end{pmatrix} + \cdots + \frac{1}{n!}\begin{pmatrix} s^n & 0 \\ 0 & t^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} s^2/2! & 0 \\ 0 & t^2/2! \end{pmatrix}$$

$$+ \begin{pmatrix} s^3/3! & 0 \\ 0 & t^3/3! \end{pmatrix} + \cdots + \begin{pmatrix} s^n/n! & 0 \\ 0 & t^n/n! \end{pmatrix}$$

$$= \begin{pmatrix} 1 + s + s^2/2! + s^3/3! + \cdots + s^n/n! & 0 \\ 0 & 1 + t + t^2/2! + t^3/3! + \cdots + t^n/n! \end{pmatrix}.$$

Letting $n \to \infty$, we find that

$$\exp(m) = \lim_{n\to\infty}\left(I + m + \frac{m^2}{2!} + \cdots + \frac{m^n}{n!}\right)$$

$$= \begin{bmatrix} \lim_{n\to\infty}(1 + s + \cdots + s^n/n!) & 0 \\ 0 & \lim_{n\to\infty}(1 + t + \cdots + t^n/n!) \end{bmatrix} = \begin{bmatrix} e^s & 0 \\ 0 & e^t \end{bmatrix}.$$

Thus,

$$\exp\left[\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} e^s & 0 \\ 0 & e^t \end{pmatrix}.$$

EXAMPLE 3. Find $\exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^n = 0$ for $n = 2, 3, \ldots$. So

$$\exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let A be a linear transformation. We define exp(A) as the linear transformation whose matrix is $\exp[m(A)]$.

EXAMPLE 4. Let $R_{\pi/2}$ be rotation by $\pi/2$. Find $\exp(R_{\pi/2})$.

Set $m = m(R_{\pi/2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$m^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)I.$$

Hence, for every positive integer k,

$$m^{2k} = ((-1)I)^k = (-1)^k I^k = (-1)^k I,$$

and so

$$m^{2k+1} = m^{2k}m = \left((-1)^k I\right)m = (-1)^k m.$$

So

$$m^3 = (-1)m, \quad m^4 = I, \quad m^5 = m, \quad m^6 = (-1)I, \quad m^7 = (-1)m,$$

and so on. Hence,

$$\exp(m) = I + m + \frac{1}{2!}(-1)I + \frac{1}{3!}(-1)m + \frac{1}{4!}I$$

$$+ \frac{1}{5!}m + \frac{1}{6!}(-1)I + \frac{1}{7!}(-1)m + \cdots$$

$$= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right)I + \left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)m.$$

We can simplify this formula by recalling that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

so

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots,$$

and

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

So

$$\exp(m) = (\cos 1)I + (\sin 1)m = (\cos 1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (\sin 1)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}\cos 1 & -\sin 1\\ \sin 1 & \cos 1\end{pmatrix}.$$

So $\exp(R_{\pi/2})$ is the linear transformation whose matrix is

$$\begin{pmatrix}\cos 1 & -\sin 1\\ \sin 1 & \cos 1\end{pmatrix}.$$

Exercise 2. Fix a scalar t and consider the matrix $\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix}$. Show that

$$\exp\left[\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix}\right] = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}. \tag{10}$$

Exercise 3. Set $m = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

(i) Calculate m^k for $k = 2, 3, 4, \ldots$.
(ii) Calculate $\exp(m)$ and simplify.

Exercise 4. Set $m = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$.

(i) Calculate m^k for $k = 2, 3, 4, \ldots$.
(ii) Calculate $\exp(m)$ and simplify.

In Chapter 2.7, we considered a linear transformation A having eigenvalues t_1, t_2 with $t_1 \neq t_2$ and eigenvectors $\mathbf{X}_1 = \begin{pmatrix}x_1\\ y_1\end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix}x_2\\ y_2\end{pmatrix}$. We defined linear transformations P and D with

$$m(P) = \begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}, \qquad m(D) = \begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix},$$

and we showed, in formula (5) of Chapter 2.7, that

$$(m(A))^n = m(P)\begin{pmatrix}t_1^n & 0\\ 0 & t_2^n\end{pmatrix}m(P^{-1}), \qquad n = 1, 2, 3, \ldots.$$

It follows that

$$\exp(m(A)) = I + m(A) + \frac{1}{2!}(m(A))^2 + \cdots$$

$$= I + m(P)\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}m(P^{-1}) + \frac{m(P)}{2!}\begin{pmatrix}t_1^2 & 0\\ 0 & t_2^2\end{pmatrix}m(P^{-1}) + \cdots$$

$$= m(P)\left[I + \begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}t_1^2 & 0\\ 0 & t_2^2\end{pmatrix} + \cdots\right]m(P^{-1})$$

(where we have used that $m(P)\cdot m(P^{-1}) = I$)

$$= m(P)\begin{bmatrix}1 + t_1 + (1/2!)t_1^2 + \cdots & 0\\ 0 & 1 + t_2 + (1/2!)t_2^2 + \cdots\end{bmatrix}m(P^{-1})$$

$$= m(P)\begin{pmatrix}e^{t_1} & 0\\ 0 & e^{t_2}\end{pmatrix}m(P^{-1}).$$

Thus, we have shown:

Theorem 2.14.

$$\exp(m(A)) = m(P)\begin{pmatrix}e^{t_1} & 0\\ 0 & e^{t_2}\end{pmatrix}m(P^{-1}). \tag{11}$$

EXAMPLE 5. Calculate $\exp\left[\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\right]$.

Here

$$t_1 = 5, \qquad t_2 = -5, \qquad \mathbf{X}_1 = \begin{pmatrix}2\\ 1\end{pmatrix}, \qquad \mathbf{X}_2 = \begin{pmatrix}-1\\ 2\end{pmatrix}.$$

So

$$m(P) = \begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}, \qquad m(P^{-1}) = \begin{pmatrix}2/5 & 1/5\\ -1/5 & 2/5\end{pmatrix}.$$

By (11), we have

$$\exp\left[\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\right] = \begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}e^5 & 0\\ 0 & e^{-5}\end{pmatrix}\begin{pmatrix}2/5 & 1/5\\ -1/5 & 2/5\end{pmatrix}$$

$$= \begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}(2/5)e^5 & (1/5)e^5\\ -(1/5)e^{-5} & (2/5)e^{-5}\end{pmatrix}$$

$$= \begin{pmatrix}(4/5)e^5 + (1/5)e^{-5} & (2/5)e^5 - (2/5)e^{-5}\\ (2/5)e^5 - (2/5)e^{-5} & (1/5)e^5 + (4/5)e^{-5}\end{pmatrix}.$$

EXAMPLE 6. Calculate $\exp\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]$.

Since $\begin{vmatrix} 1-t & 0 \\ 0 & -t \end{vmatrix} = t^2 - t = t(t-1)$, the eigenvalues are $t_1 = 1$, $t_2 = 0$. The corresponding eigenvectors are $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{X}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $m(P) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, and then $m(P^{-1}) = I$. Hence, by (11),

$$\exp\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = I\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}I = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 5.

(a) Compute $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^n$ for $n = 1, 2, 3, \ldots$.

(b) Compute $\exp\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$ directly from the definition and compare your answer with the result of Example 6.

Exercise 6. Using Theorem 2.14, calculate $\exp\left[\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}\right]$.

Exercise 7. Calculate $\exp\left[\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\right]$.

Exercise 8. Calculate $\exp\left[\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right]$.

Recall Eq. (5): $d\mathbf{X}/dt = m(\mathbf{X}(t))$, where $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We fix a vector $\mathbf{X}$ and define $\mathbf{X}(t) = \exp(tm)(\mathbf{X}_0)$. In §2, we shall show that $\mathbf{X}(t)$ solves (5) and satisfies the initial condition $\mathbf{X}(0) = \mathbf{X}_0$, and we shall study examples and applications.

§2. Solutions of Differential Systems

We fix a matrix m and a vector $\mathbf{X}_0$.

$$\exp(tm) = I + tm + \frac{t^2}{2!}m^2 + \frac{t^3m^3}{3!} + \cdots,$$

so

$$(\exp(tm))(\mathbf{X}_0) = \mathbf{X}_0 + tm(\mathbf{X}_0) + \frac{t^2}{2!}m^2(\mathbf{X}_0) + \frac{t^3}{3!}m^3(\mathbf{X}_0) + \cdots.$$

Both sides of the last equation are vector-valued functions of t. It can be shown that the derivative of the sum of the infinite series is obtained by differentiating the series term by term. In other words,

$$\frac{d}{dt}\{(\exp(tm))(\mathbf{X}_0)\} = \frac{d}{dt}(tm(\mathbf{X}_0)) + \frac{d}{dt}\left(\frac{t^2}{2!}m^2(\mathbf{X}_0)\right) + \cdots. \quad (12)$$

The right-hand side of (12) is equal to

$$m(\mathbf{X}_0) + \frac{2t}{2!}m^2(\mathbf{X}_0) + \frac{3t^2}{3!}m^3(\mathbf{X}_0) + \frac{4t^3}{4!}m^4(\mathbf{X}_0) + \cdots$$
$$= m(\mathbf{X}_0) + tm^2(\mathbf{X}_0) + \frac{t^2}{2!}m^3(\mathbf{X}_0) + \frac{t^3}{3!}m^4(\mathbf{X}_0) + \cdots$$
$$= m(\mathbf{X}_0) + m(tm(\mathbf{X}_0)) + m\left(\frac{t^2}{2!}m^2(\mathbf{X}_0)\right) + m\left(\frac{t^3}{3!}m^3(\mathbf{X}_0)\right) + \cdots$$
$$= m\left\{\mathbf{X}_0 + tm(\mathbf{X}_0) + \frac{t^2}{2!}m^2(\mathbf{X}_0) + \frac{t^3}{3!}m^3(\mathbf{X}_0) + \cdots\right\}$$
$$= m\{(\exp(tm))(\mathbf{X}_0)\}.$$

So (12) gives us

$$\frac{d}{dt}\{(\exp(tm))(\mathbf{X}_0)\} = m\{(\exp(tm))(\mathbf{X}_0)\}. \quad (13)$$

We define $\mathbf{X}(t) = (\exp(tm))(\mathbf{X}_0))$. Then (13) states that

$$\frac{d\mathbf{X}}{dt}(t) = m(\mathbf{X}(t)). \quad (14)$$

In other words, we have shown that $\mathbf{X}(t)$ solves our original equation (5). Also, setting $t = 0$ in the definition of $\mathbf{X}(t)$, we find that

$$\mathbf{X}(0) = I(\mathbf{X}_0) = \mathbf{X}_0, \quad (15)$$

since $\exp(0) = I + 0 + 0 + \cdots = I$. So we have proved:

Theorem 2.15. *Let m be a matrix. Fix a vector $\mathbf{X}_0$. Set $\mathbf{X}(t) = (\exp(tm))(\mathbf{X}_0)$ for all t. Then,*

$$\frac{d\mathbf{X}}{dt} = m\mathbf{X}(t), \quad (16)$$

and

$$\mathbf{X}(0) = \mathbf{X}_0. \quad (17)$$

EXAMPLE 7. Solve the differential system

$$\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \end{cases} \quad (18)$$

with the initial condition: $x(0) = 1$, $y(0) = 0$.

In vector form, with $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we have

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(\mathbf{X})$$

with initial condition $\mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Set $m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{X}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and set

$$\mathbf{X}(t) = \exp(tm)(\mathbf{X}_0).$$

By Exercise 2 in this chapter,

$$\exp(tm) = \exp\left[\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

So

$$\mathbf{X}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Since $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we obtain $x(t) = \cos t$, $y(t) = \sin t$. Inserting these functions in (18), we see that it checks. Also, $x(0) = 1$, $y(0) = 0$, so the initial condition checks also.

EXAMPLE 8. Solve the differential system (18) with initial condition $x(0) = x_0, y(0) = y_0$.

We take $\mathbf{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and set

$$\mathbf{X}(t) = \left(\exp\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}\right)(\mathbf{X}_0) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

so

$$\mathbf{X}(t) = \begin{pmatrix} (\cos t)x_0 - (\sin t)y_0 \\ (\sin t)x_0 + (\cos t)y_0 \end{pmatrix}.$$

So

$$x(t) = (\cos t)x_0 - (\sin t)y_0, \qquad y(t) = (\sin t)x_0 + (\cos t)y_0.$$

We check that these functions satisfy (18) and that $x(0) = x_0, y(0) = y_0$.

Exercise 9. Calculate $\exp\left[\begin{pmatrix} 3t & 4t \\ 4t & -3t \end{pmatrix}\right]$, where t is a given number.

Exercise 10. Using the result of Exercise 9, solve the system

$$\begin{cases} \dfrac{dx}{dt} = 3x + 4y \\ \dfrac{dy}{dt} = 4x - 3y \end{cases} \quad \text{with} \quad \begin{Bmatrix} x(0) = 1 \\ y(0) = 0 \end{Bmatrix}, \tag{19}$$

by using Theorem 2.15 with $m = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ and $\mathbf{X}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Exercise 11. Solve the system (19) with $x(0) = s_1, y(0) = s_2$.

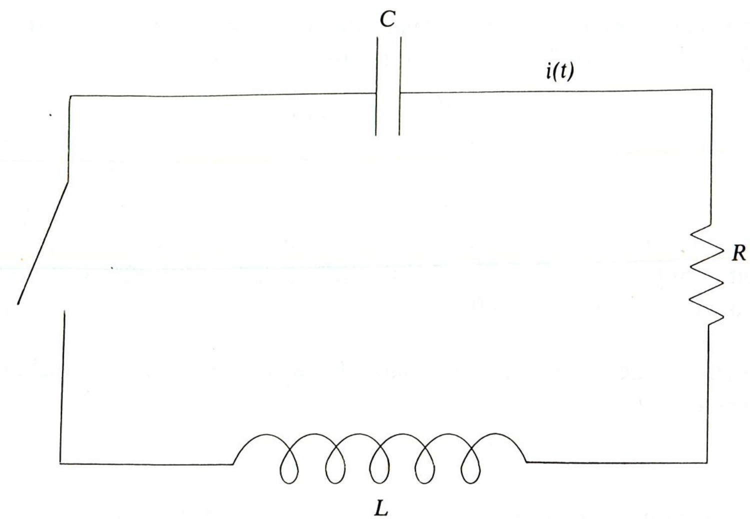

Figure 2.53

Exercise 12. Solve the system

$$\begin{cases} \dfrac{dx}{dt} = 2x + 4y \\ \dfrac{dy}{dt} = 4x + 6y \end{cases} \quad \text{with} \quad \begin{Bmatrix} x(0) = 1 \\ y(0) = 0 \end{Bmatrix}. \tag{20}$$

Exercise 13. Solve the system (1) (at the beginning of this chapter) with initial condition (2).

EXAMPLE 9. Consider an electric circuit consisting of a condenser of capacitance C connected to a resistance of R ohms and an inductance of L henries. A switch is inserted in the circuit (see Fig. 2.53). The condenser is charged with a charge of Q_0 coulombs, with the switch open. At time $t = 0$, the switch is closed and the condenser begins to discharge, causing a current to flow in the circuit. Denote by $i(t)$ the current flowing at time t and by $Q(t)$ the charge on the condenser at time t. The laws of electricity tell us the following: the voltage drop at time t equals $(1/C)Q(t)$ across the condenser, while the voltage drop across the resistance is $Ri(t)$ and the voltage drop across the inductance is $L(di/dt)$. The sum of all the voltage drops equals 0 at every time $t > 0$, since the circuit is closed. Thus, we have

$$\frac{1}{C}Q(t) + Ri(t) + L\frac{di}{dt} = 0$$

or

$$\frac{di}{dt} = -\frac{1}{LC}Q(t) - \frac{R}{L}i(t).$$

Also, the current at time t equals the negative of dQ/dt or $i(t) = -dQ/dt$. So the two functions: $t \to i(t)$ and $t \to Q(t)$ satisfy

$$\begin{cases} \dfrac{di}{dt} = ai + bQ, \\ \dfrac{dQ}{dt} = -i, \end{cases} \tag{21}$$

where $a = -R/L$, $b = -1/LC$. So to calculate the current flowing in the circuit at any time t, we must solve the differential system (21) with initial condition $Q(0) = Q_0$, $i(0) = 0$.

EXAMPLE 10. Let c_1, c_2 be two scalars. We wish to solve the *second-order differential equation*

$$\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} + c_2x = 0 \tag{22}$$

by a function $t \to x(t)$ defined for all t, and we want to satisfy the initial conditions

$$x(0) = x_0, \qquad \frac{dx}{dt}(0) = y_0. \tag{23}$$

We shall reduce the problem (22) to a *first-order differential system* of the form (3). To this end we define $y(t) = (dx/dt)(t)$. Then (22) can be written: $dy/dt + c_1y + c_2x = 0$ or

$$\frac{dy}{dt} = -c_2x - c_1y.$$

So x and y satisfy the differential system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -c_2x - c_1y. \end{cases} \tag{24}$$

EXAMPLE 11. We study the equation

$$\frac{d^2x}{dt^2} + x = 0, \qquad x(0) = x_0, \qquad \frac{dx}{dt}(0) = y_0. \tag{25}$$

Setting $y = dx/dt$, (25) turns into

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x, \end{cases} \qquad x(0) = x_0, \quad y(0) = y_0. \tag{26}$$

Exercise 14. Fix a scalar t. Show that

$$\exp\left[t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Exercise 15. Using the result of Exercise 14, solve first the equations (26) and then Eq. (25).

In Theorem 2.15, we showed that the problem $d\mathbf{X}/dt = m\mathbf{X}(t)$, $\mathbf{X}(0) = \mathbf{X}_0$ has $\mathbf{X}(t) = (\exp(tm)(\mathbf{X}_0))$ as a solution for all t. We shall now show that this is the *only* solution, or, in other words, we shall prove uniqueness of solutions.

Suppose $\mathbf{X}$, $\mathbf{Y}$ are two solutions. Then $d\mathbf{X}/dt = m\mathbf{X}(t)$, $\mathbf{X}(0) = \mathbf{X}_0$ and $d\mathbf{Y}/dt = m\mathbf{Y}(t)$, $\mathbf{Y}(0) = \mathbf{X}_0$. Set $\mathbf{Z}(t) = \mathbf{X}(t) - \mathbf{Y}(t)$. Our aim is to prove that $\mathbf{Z}(t) = 0$ for all t. We have

$$\begin{aligned} \frac{d\mathbf{Z}}{dt} &= \frac{d\mathbf{X}}{dt} - \frac{d\mathbf{Y}}{dt} = m\mathbf{X}(t) - m\mathbf{Y}(t) \\ &= m(\mathbf{X}(t) - \mathbf{Y}(t)) = m\mathbf{Z}(t). \end{aligned} \tag{27}$$

Also

$$\mathbf{Z}(0) = \mathbf{X}(0) - \mathbf{Y}(0) = \mathbf{X}_0 - \mathbf{X}_0 = 0. \tag{28}$$

We now shall use (27) and (28) to show that $\mathbf{Z}(t) = \mathbf{0}$ for all t. We denote by $f(t)$ the squared length of $\mathbf{Z}(t)$, i.e.,

$$f(t) = |\mathbf{Z}(t)|^2.$$

$t \to f(t)$ is a scalar-valued function. It satisfies

$$f(t) \geqslant 0 \qquad \text{for all } t \quad \text{and} \quad f(0) = 0.$$

Exercise 16. If $\mathbf{A}(t)$, $\mathbf{B}(t)$ are two vector-valued functions, then

$$\frac{d}{dt}(\mathbf{A}(t)\cdot\mathbf{B}(t)) = \mathbf{A}(t)\cdot\frac{d\mathbf{B}}{dt} + \mathbf{B}(t)\cdot\frac{d\mathbf{A}}{dt}.$$

It follows from Exercise 16 that

$$\frac{df}{dt} = \frac{d}{dt}(\mathbf{Z}(t)\cdot\mathbf{Z}(t)) = \mathbf{Z}(t)\cdot\frac{d\mathbf{Z}}{dt} + \mathbf{Z}(t)\cdot\frac{d\mathbf{Z}}{dt} = 2\mathbf{Z}(t)\cdot\frac{d\mathbf{Z}}{dt}.$$

Using (27), this gives

$$\frac{df}{dt}(t) = 2\mathbf{Z}(t)\cdot m\mathbf{Z}(t). \tag{29}$$

We set $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Fix t and set $\mathbf{Z}(t) = \mathbf{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then

$$\begin{aligned} 2\mathbf{Z}(t)\cdot m\mathbf{Z}(t) &= 2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\cdot\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right\} \\ &= 2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\cdot\begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} \\ &= 2(az_1^2 + bz_1z_2 + cz_2z_1 + dz_2^2). \end{aligned}$$

Let K be a constant greater than $|a|$, $|b|$, $|c|$, $|d|$. Then

$$|2\mathbf{Z}(t)\cdot m\mathbf{Z}(t)| \leqslant 2(|a|z_1^2 + |b|\,|z_1|\,|z_2| + |c|\,|z_2|\,|z_1| + |d|\,|z_2|^2)$$
$$\leqslant 2K(|z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2).$$

Also,

$$2|z_1|\,|z_2| \leqslant |z_1|^2 + |z_2|^2.$$

So

$$|2\mathbf{Z}(t)\cdot m\mathbf{Z}(t)| \leqslant 2K(2|z_1|^2 + 2|z_2|^2) = 4K(|z_1|^2 + |z_2|^2)$$
$$= 4K|\mathbf{Z}|^2 = 4Kf(t).$$

By (29), setting $M = 4K$, this gives

$$\frac{df}{dt}(t) \leqslant Mf(t). \tag{30}$$

Consider the derivative

$$\frac{d}{dt}\left(\frac{f(t)}{e^{Mt}}\right) = \frac{e^{Mt}(df/dt) - f(t)Me^{M(t)}}{e^{2Mt}} = \frac{(df/dt) - Mf(t)}{e^{Mt}}.$$

By (30), the numerator of the right-hand term $\leqslant 0$ for all t. So

$$\frac{d}{dt}\left(\frac{f(t)}{e^{Mt}}\right) \leqslant 0,$$

so $f(t)/e^{Mt}$ is a decreasing function of t. Also, $f(t)/e^{Mt} \geqslant 0$ and $= 0$ at $t = 0$. But a decreasing function of t, defined on $t \geqslant 0$ which is $\geqslant 0$ for all t and $= 0$ at $t = 0$, is identically 0.

So $f(t)/e^{Mt} = 0$ for all t. Thus $|\mathbf{Z}(t)|^2 = f(t) = 0$, and so $\mathbf{Z}(t) = 0$, and so $\mathbf{X}(t) = \mathbf{Y}(t)$ for all t.

We have proved:

Uniqueness Property. *The only solution of the problem considered in Theorem 2.15 is* $\mathbf{X}(t) = (\exp(tm))(\mathbf{X}_0)$.