There's not much to this one, at least to the mandatory part.

6 For this problem, you first need to show the vectors are parallel. There are probably three ways to do this. One is to notice that they all point in the y-direction. One is to compute the cross product of two arbitrary vectors, and one is to show that two arbitrary vectors are scalar multiples.

The second part is to compute the curl. That is merely computation. The text says that the curl will not be zero.

The third part is to explain why the leaf still turns. Because the problem _says_ that the curl is not zero, that is not enough reason. The reason must be something like the forces on one side are stronger than on the other.

7 Compute the derivative of the differential form. _Please_ do not use a formula to compute the derivative. At this point you really need to be able to compute derivatives of differential forms. Beyond that, then you compute a line integral. You also need to know how to compute line integrals.

The important point of this problem is that differential forms that are closed in simply connected regions are exact, but this one is closed in three-space without the z-axis, which is not simply connected. If it were exact then the integral around any closed loop would be zero.

9 - 10 These problems are the same. They both involve finding anti-derivatives. Probably the best way is to integrate the three components separately and then hope that they fit together. Notice for 10 that ln(r) and 1/2ln(r^2) are the same.

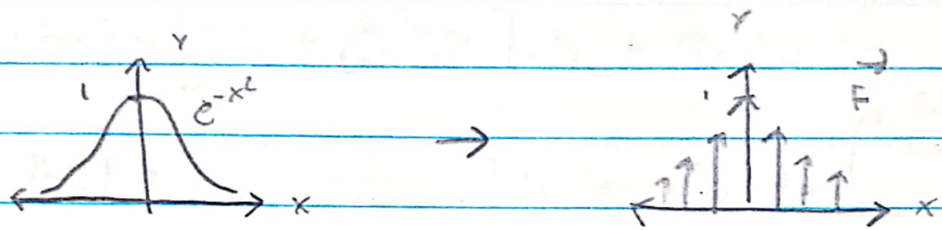
One student submitted two problems for the fake problem set.

11.4 8 Involves using the referenced equation, making one substitution, then seeing what terms you can eliminate by using physics version of Maxwell's equations. There is a small last step of recognising the derivative of x^2 is 2xdx.

11.4 13 is a valuable exercise of cycling through the different forms of Maxwell's equations as differential forms. This is the crux is 11.3.

10.8

⑥ $\vec{F} = (0, e^{-x^2}, 0)$



2D projection

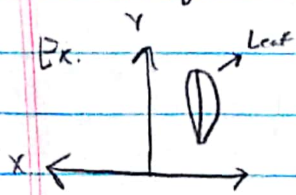
Vectors continue into $\hat{z}$ direction

Since $\vec{F}$ only points in the $\hat{y}$ direction $(\vec{F} = F\hat{y})$ all vectors are parallel

$$\text{Curl} = \vec{\nabla} \times \vec{F} = \left(0 - \frac{\partial}{\partial z}(e^{-x^2}), 0-0, \frac{\partial}{\partial x}(e^{-x^2}) - 0\right)$$
$$= (0, 0, e^{-x^2}(-2x)) \neq (0,0,0)$$

If a leaf was perfectly symmetric about an axis and that axis was lined up exactly with the $\hat{y}$ axis, then the leaf would be in an unstable equillibrium and wouldn't rotate, also assuming the leaf is "2D" and flush with the surface of the water. However, if all of those conditions are not met, the leaf would experience an uneven force across it, causing a torque about some pivot, causing it to rotate

Ex.



More force on left side, causes a clockwise rotation

7) $w = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} + 1dz$

· show $dw = 0$

$$dw = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}\, dydx + \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\, dxdy$$

$$= \frac{(x^2+y^2) - (2y^2) + (x^2+y^2) - (2x^2)}{(x^2+y^2)^2}\, dxdy$$

$$= \frac{2x^2 + 2y^2 - 2y^2 - 2x^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = \boxed{0 = dw}$$

Evaluate:

$$\oint_C \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} + dz \qquad C = \{(\cos\theta, \sin\theta, 0) \mid 0 \le \theta \le 2\pi\}$$

$x = \cos\theta \quad dx = -\sin\theta \, d\theta$

$y = \sin\theta \quad dy = \cos\theta \, d\theta$ • replace x, y, z, dx, dy, dz

$z = 0 \quad dz = 0$

$$\oint w = \int_0^{2\pi} \frac{+\sin\theta}{(\sin^2\theta + \cos^2\theta)^2} \sin\theta \, d\theta + \frac{\cos\theta}{(\sin^2\theta + \cos^2\theta)^2} \cos\theta \, d\theta + 0$$

$$= \int_0^{2\pi} \frac{\sin^2\theta + \cos^2\theta}{(\sin^2\theta + \cos^2\theta)^2} \, d\theta = \int_0^{2\pi} d\theta \quad ①$$

① • Because $\oint w = \int^{2\pi} d\theta = 2\pi$ is path dependent (ie. $\int d\theta$ depends on the bands), w is <u>not</u> exact

9. Find a potential field for

$$w = r^a(x\,dx + y\,dy + z\,dz), \quad a \neq -2.$$

w can be rewritten as

$$w = \underbrace{r^a x\,dx}_{①} + \underbrace{r^a y\,dy}_{②} + \underbrace{r^a z\,dz}_{③}.$$

We will find the potential field by separating w into 3 parts.

let $t = x^2+y^2+z^2$.

$dt = 2x\,dx$

$\frac{dt}{2} = x\,dx$.

$$① = \int r^a x\,dx = \int x\sqrt{x^2+y^2+z^2}^{\,a}\,dx = \int x(x^2+y^2+z^2)^{a/2}\,dx.$$

$$= \int t^{a/2} \cdot \frac{1}{2}\,dt.$$

$$= \frac{t^{\frac{a}{2}+1}}{\frac{a}{2}+1} \cdot \frac{1}{2} + g(y,z)$$ where g is some function or constant of y and z.

$$= \boxed{\frac{t^{\frac{a}{2}+1}}{a+2} + g(y,z)}$$ → We will call this Ⓢ

If we take partial respective to y & z, we should get ② and ③.

Well, maybe not exactly ②, ③, but something similar enough to find the exact functions of y and z.

If we take a derivative respect to y, we get.

$$\frac{\partial \textcircled{f}}{\partial y} = \frac{(\frac{a}{2}+1)\, t^{a/2} \cdot 2y}{(2+a)} + \frac{\partial g}{\partial y} = t^{a/2} y + \frac{\partial g}{\partial y}.$$

since ② $= r^a y dy$, $\frac{\partial g}{\partial y} = 0$.

Next, we solve for ③, $r^a z dz$.

$$\frac{\partial \textcircled{f}}{\partial z} = \frac{(\frac{a}{2}+1)\, t^{a/2}\, 2 \cdot z}{(2+a)} + \frac{\partial g}{\partial z} = t^{a/2} z + \frac{\partial g}{\partial z}.$$

since ③ $= r^a z dz$, $\frac{\partial g}{\partial z} = 0$.

Therefore, we can conclude that $g(y,z)$ is just some constant.

$\longrightarrow$ potential field: $\boxed{\dfrac{(x^2+y^2+z^2)^{\frac{a}{2}+1}}{a+2} + C. \quad \text{where } a \neq -2.}$

10. Let $\omega = \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2}$. We must find a scalar field ϕ such that $\omega = d\phi$.

Since $\omega = \frac{x}{x^2+y^2+z^2}dx + \frac{y}{x^2+y^2+z^2}dy + \frac{z}{x^2+y^2+z^2}dz$, let's compute

$\int \frac{x}{x^2+y^2+z^2}dx$. Let $u = x^2+y^2+z^2 \rightarrow du = 2x\,dx$

$\frac{1}{2x}du = dx$

$$\int \frac{x}{x^2+y^2+z^2}dx = \int \frac{x}{u}\left(\frac{1}{2x}du\right) = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(x^2+y^2+z^2)$$

Since the second and third terms are of the same form, we see easily that they will both give $\frac{1}{2}\ln(x^2+y^2+z^2)$.

Potential field: $\boxed{\phi = \frac{1}{2}\ln(x^2+y^2+z^2)}$

Homework #7:

#8. Using equation

$$\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})$$

and maxwell's equation, prove that

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}E^2 + B^2\right) = 2\nabla\cdot(\vec{B}\times\vec{E}) - 2\mu\vec{J}\cdot\vec{E}$$

where $E^2 = \vec{E}\cdot\vec{E}$ and $B^2 = \vec{B}\cdot\vec{B}$

looking at the right hand side of the equation

$$2\nabla\cdot(\vec{B}\times\vec{E}) - 2\mu\vec{J}\cdot\vec{E}$$

we will use $\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})$

Then plugging in,

$$2\nabla\cdot(\vec{B}\times\vec{E}) - 2\mu\vec{J}\cdot\vec{E} = 2(\nabla\times\vec{B})\cdot\vec{E} - 2\vec{B}\cdot(\nabla\times\vec{E}) - 2\mu\vec{J}\cdot\vec{E}$$

Then using maxwell's equations:

$$\nabla\cdot\vec{B} = 0 \quad , \quad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\epsilon\nabla\cdot\vec{E} = \rho \quad , \quad \nabla\times\vec{B} - \epsilon\mu\frac{\partial\vec{E}}{\partial t} = \mu\vec{J}$$

Thus

$$2(\nabla\times\vec{B})\cdot\vec{E} - 2\vec{B}\cdot(\nabla\times\vec{E}) - 2\mu\vec{J}\cdot\vec{E} = 2\left(\mu\vec{J} + \epsilon\mu\frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E} - 2\vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - 2\mu\vec{J}\cdot\vec{E}$$

$$= 2\mu\vec{J}\cdot\vec{E} + 2\epsilon\mu\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + 2\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - 2\mu\vec{J}\cdot\vec{E} = +2\epsilon\mu\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + 2\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(\epsilon\mu\,\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) = \frac{\partial}{\partial t}\left(-\epsilon\mu E^2 + B^2\right)$$

We know $c = \frac{1}{\sqrt{\epsilon\mu}}$ thus $\epsilon\mu = \frac{1}{c^2}$

$$= \frac{\partial}{\partial t}\left(-\frac{1}{c^2}E^2 + B^2\right)$$

Therefore, we have solved for what u expected □

#13 $\Phi = x^2\cos(z-ct)dxdydz - cx^2\cos(z-ct)dxdydt$

We want to verify that Φ is closed. The definition of closed is $d\Phi = 0$

$d\Phi = cx^2\sin(z-ct)dtdxdydz + cx^2\sin(z-ct)dzdxdydt$
$= -cx^2\sin(z-ct)dxdydzdt + cx^2\sin(z-ct)dxdydzdt = 0$

Therefore, Φ is closed.

Our goal here is to turn this 3-form into a 1-form. The equation for a 3-form in $\mathbb{R}^4$ is:

$$\Phi = A_4\, dxdydz + \frac{1}{\epsilon\mu}A_1 dydzdt + \frac{1}{\epsilon\mu}A_2 dzdxdt + \frac{1}{\epsilon\mu}A_3 dxdydt$$

From this equation, we can determine the A values:

$A_4 = x^2\cos(z-ct)$ $\quad A_3 = -\epsilon\mu c x^2\cos(z-ct)$

$A_1 = 0$ $\quad A_2 = 0$

Using our equation of a 1-form,

$$\Phi = A_1 dx + A_2 dy + A_3 dz + A_4 dt$$

we can plug in our values of A.

$$\Phi = -\epsilon\mu c x^2\cos(z-ct)dz + x^2\cos(z-ct)dt$$

From this equation, we can determine B : E.

$B + Edt = -d\Phi = -d\left(-\epsilon\mu c x^2\cos(z-ct)dz + x^2\cos(z-ct)dt\right)$
$= -\left[-\epsilon\mu c 2x\cos(z-ct)dxdz - \epsilon\mu c^2x^2\sin(z-ct)dtdz + 2x\cos(z-ct)dxdt\right.$
$\left. - x^2\sin(z-ct)dzdt\right] = -\epsilon\mu c 2x\cos(z-ct)dzdx - \epsilon\mu c^2x^2\sin(z-ct)dzdt$
$-2x\cos(z-ct)dxdt + x^2\sin(z-ct)dzdt = -\epsilon\mu c 2x\cos(z-ct)dzdx$
$-2x\cos(z-ct)dxdt$

Since $\epsilon\mu = \frac{1}{c^2}$ $\quad B + Edt = -c^{-1}2x\cos(z-ct)dzdx - 2x\cos(z-ct)dxdt$

using the equation

$$B + E\,dt = B_1 dy dz + B_2 dz dx + B_3 dx dy + E_1 dx dt + E_2 dy dt + E_3 dz dt$$

From this equation, we can determine $\vec{B}$; $\vec{E}$

$B = -2xc^{-1}\cos(z-ct)\,dz dx \Rightarrow \vec{B} = (0, -2xc^{-1}\cos(z-ct), 0)$

$E = -2x\cos(z-ct)\,dx \Rightarrow \vec{E} = (-2x\cos(z-ct), 0, 0)$

We need to find $\vec{J}$; ρ. We use the equation

$$\frac{1}{\epsilon} J = d\left(E - \frac{1}{\epsilon\mu} B\,dt\right)$$

First finding (did this by switching space time units of B ; E)

$E - \frac{1}{\epsilon\mu} B\,dt = -2x\cos(z-ct)\,dy dz + 2cx\cos(z-ct)\,dy dt$

$\frac{1}{\epsilon} J = d\left(E - \frac{1}{\epsilon\mu} B\,dt\right) = -2\cos(z-ct)\,dx dy dz + 2cx\sin(z-ct)\,dt dy dz$

$-2c\cos(z-ct)\,dx dy dt + 2cx\cos(z-ct)\,dz dy dt$

$= -2\cos(z-ct)\,dx dy dz + 2cx\sin(z-ct)\,dy dz dt - 2c\cos(z-ct)\,dx dy dt$

$-2cx\cos(z-ct)\,dy dz dt = -2\cos(z-ct)\,dx dy dz - 2c\cos(z-ct)\,dx dy dt$

$J = -2\epsilon\cos(z-ct)\,dx dy dz - 2c\epsilon\cos(z-ct)\,dx dy dt$

using the equation:

$$J = \rho\,dx dy dz - J_1 dy dz dt - J_2 dz dx dt - J_3 dx dy dt$$

$\rho = -2\epsilon\cos(z-ct)$

$\vec{J} = (0, 0, -2c\epsilon\cos(z-ct))$