Comments and student work for PS4

8.3 9. was mostly straightforward. Everyone did fine with it.

10-12 was the interesting part of the problem set. There are two errors in this sequence of problems in the book. One is conceptual and disturbs me more, and one is quantitative and maybe disturbs you more. The conceptual error is that the entire problem refers to an n-sphere. The 1-sphere is a circle, the 2-sphere is the surface of a beach ball. This is *not* what it intended. What is intended is an n-ball. The 2-ball is the inside surface of a disc, the 3-ball is a solid filled in ball like a billiard ball.

The quantitative error is in the clue in the back for 12. The factor in front is the reciprocal of the correct factor. It should be $\frac{n-1}{n}$. This is easily verified by asking a machine to compute the integral for $n = 3$. The integral for $n = 1$ gives a value of 2. If you multiply that by 3/2 you get 3, but the integral for $n = 3$ is 4/3. Notice multiplying 2 by 2/3 gives the correct answer.

8.3.10 Explaining the pattern is important, as is showing that these coordinates satisfy the equation for an n-sphere (here I mean sphere) of radius ρ. The ball is the collection of all spheres from radius 0 to r. If you do not explain why we need to extend our coordinates each step in this way you have shown that the points are some of the points in a ball, but not necessarily all of them. It is interesting that you have known the equation for a n-sphere for a long time because you see how to generalise from the circle to the sphere.

8.3.11 This question is probably the hardest of this set. I have good news and promise to keep - I will have no more n-dimensional questions for the remainder of the course. Adding a multiple of a row to another does not change the determinant. The same is true for a column (because taking the transpose does not change the determinant).

8.3.12 This integral is greatly simplified because you are integrating over a box - all bounds for the integrals are numbers, and also because the variables are isolated in multiplication. Using the fact that $\int_a^b \int_c^d f(x)g(y)dydx = \int_a^b f(x)dx \int_c^d g(y)dy$ greatly simplifies this computation.

Therefore $\int_{S(r)} dx_1 dx_2 \dots dx_n = \int_B \rho^{n-1} \cos\phi_1 \cos^2\phi_2 \dots \cos^{n-2}\phi_{n-2} d\rho d\theta d\phi_1 d\phi_2 \dots d\phi_{n-2}$
$= \int_0^r \rho^{n-1} d\rho \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\phi_1 d\phi_1 \int_{-\pi/2}^{\pi/2} \cos^2\phi_2 d\phi_2 \dots \int_{-\pi/2}^{\pi/2} \cos^{n-2}\phi_{n-2} d\phi_{n-2}$

Using the above clue, and the easy first two integrals, you can build the pattern given in the text.

8.6.5 Please remember you need a justification for orientation - how you know to pull the form back to $d\theta dz$ and not $dz d\theta$. This is done in the exemplar. Without it, your work is only a lucky guess.

8.6.11 Is merely computing $d\sigma$ using the definition $d\sigma = \left|\frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta}\right| dr d\theta$. In this case, F is the function from (r, θ) to (x, y, z). There is no orientation check needed here because of the magnitude.

8.6.13 it seems the biggest challenge in a problem that was mostly routine was to understand that a torus is made by rotating a circle around another circle. Therefore the limits for both angle variables need to include a *full* circle. Most likely this means both angles go from 0 to 2π, but any range that includes the full circle, e.g. $-\pi$ to π would suffice.

8.3

9. $x = \rho\cos\theta\cos\phi$

$y = \rho\sin\theta\cos\phi$

$z = \rho\sin\phi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\cos\phi & -\rho\cos\phi\sin\theta & -\rho\cos\theta\sin\phi \\ \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \end{vmatrix}$$

$$J = \cos\theta\cos\phi\left(\rho^2\cos\theta\cos^2\phi\right) + \rho\cos\phi\sin\theta\left(\rho\sin\theta\cos^2\phi + \rho\sin\theta\sin^2\phi\right)$$
$$- \rho\cos\theta\sin\phi\left(0 - \rho\cos\theta\cos\phi\sin\phi\right)$$
$$= \rho^2\cos^2\theta\cos^3\phi + \rho^2\sin^2\theta\cos^3\phi + \rho^2\cos\phi\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta\sin^2\phi\cos\phi$$
$$= \rho^2\cos^3\phi\left(\sin^2\theta + \cos^2\theta\right) + \rho^2\sin^2\phi\cos\phi\left(\sin^2\theta + \cos^2\theta\right)$$
$$= \rho^2\cos^3\phi + \rho^2\left(1 - \cos^2\phi\right)\cos\phi = \rho^2\cos^3\phi + \rho^2\cos\phi - \rho^2\cos^3\phi$$
$$= \rho^2\cos\phi$$

10) We know that the parametrized coordinates of a $n=2$ dimensional circle with radius ρ are $x_1=\rho\cos\theta$ and $x_2=\rho\sin\theta$. To compute the coordinates of a $n=3$ dimensional sphere, we multiply x_1 by $\cos\Phi_1$, x_2 by $\cos\Phi_1$, and add an $x_{n=3}$ coordinate to find $x_1=\rho\cos\theta\cos\Phi_1$, $x_2=\rho\sin\theta\cos\Phi_1$, $x_3=\rho\sin\Phi_1$. Similarly, to compute the coordinates of a $n=4$ dimensional sphere, we multiply x_1, x_2, and x_3 by $\cos\Phi_2$ and add an $x_{n=4}$ coordinate to find $x_1=\rho\cos\theta\cos\Phi_1\cos\Phi_2$, $x_2=\rho\sin\theta\cos\Phi_1\cos\Phi_2$, $x_3=\rho\sin\Phi_1\cos\Phi_2$, $x_4=\rho\sin\Phi_2$. Finding the coordinates of a $n=5$ dimensional sphere, we multiply x_1, x_2, x_3, x_4 by $\cos\Phi_3$ and add an $x_{n=5}$ coordinate to find $x_1=\rho\cos\theta\cos\Phi_1\cos\Phi_2\cos\Phi_3$, $x_2=\rho\sin\theta\cos\Phi_1\cos\Phi_2\cos\Phi_3$, $x_3=\rho\sin\Phi_1\cos\Phi_2\cos\Phi_3$, $x_4=\rho\sin\Phi_2\cos\Phi_3$, $x_5=\rho\sin\Phi_3$. Converting back to Cartesian coordinates for our 5D sphere, our equation becomes $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=\rho^2\cos^2\theta\cos^2\Phi_1\cos^2\Phi_2\cos^2\Phi_3+\rho^2\sin^2\theta\cdot\cos^2\Phi_1\cos^2\Phi_2\cos^2\Phi_3+\rho^2\sin^2\Phi_1\cos^2\Phi_2\cos^2\Phi_3+\rho^2\sin^2\Phi_2\cos^2\Phi_3+\rho^2\sin^2\Phi_3=\rho^2$, as desired. Thus this parametrization holds.

Now consider the case of the 4-sphere, with the given parametrization

$$\begin{aligned} x_1 &= \rho \cos\theta \cos\phi_1 \cos\phi_2 \\ x_2 &= \rho \sin\theta \cos\phi_1 \cos\phi_2 \\ x_3 &= \rho \sin\phi_1 \cos\phi_2 \\ x_4 &= \rho \sin\phi_2 \end{aligned}$$

Here, the Jacobian matrix takes the form

$$\frac{d(x_1, x_2, x_3, x_4)}{d(\rho, \theta, \phi_1, \phi_2)} = \begin{bmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \phi_1} & \frac{\partial x_1}{\partial \phi_2} \\ \frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial \phi_1} & \frac{\partial x_2}{\partial \phi_2} \\ \frac{\partial x_3}{\partial \rho} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial \phi_1} & \frac{\partial x_3}{\partial \phi_2} \\ \frac{\partial x_4}{\partial \rho} & \frac{\partial x_4}{\partial \theta} & \frac{\partial x_4}{\partial \phi_1} & \frac{\partial x_4}{\partial \phi_2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi_1\cos\phi_2 & -\rho\cos\phi_1\cos\phi_2\sin\theta & -\rho\cos\theta\cos\phi_2\sin\phi_1 & -\rho\cos\theta\cos\phi_1\sin\phi_2 \\ \cos\phi_1\cos\phi_2\sin\theta & \rho\cos\theta\cos\phi_1\cos\phi_2 & -\rho\cos\phi_2\sin\theta\sin\phi_1 & -\rho\cos\phi_1\sin\theta\sin\phi_2 \\ \cos\phi_2\sin\phi_1 & 0 & \rho\cos\phi_1\cos\phi_2 & -\rho\sin\phi_1\sin\phi_2 \\ \sin\phi_2 & 0 & 0 & \rho\cos\phi_2 \end{bmatrix}$$

We subtract the product of each element in the last column and the ratio of the first element in the first column to the first element in the last column from the first column of the Jacobian matrix, all elements but the first are effectively eliminated from the first column. Similarly, by following a similar procedure for the second and third columns of the Jacobian Matrix, the first element in the second column and the first two elements in the third column become zeros:

$$\begin{bmatrix} 0 & 0 & 0 & -\rho\cos\theta\cos\phi_1\sin\phi_2 \\ 0 & \rho\sec\theta\cos\phi_1\cos\phi_2 & 0 & -\rho\cos\phi_1\sin\theta\sin\phi_2 \\ 0 & \rho\cos\phi_2\sin\phi_1\tan\theta & \rho\sec\phi_1\cos\phi_2 & -\rho\sin\phi_1\sin\phi_2 \\ \csc\phi_2 & -\rho\cos\phi_2\cot\phi_2\tan\theta & -\rho\cos\phi_2\cot\phi_2\phi_1 & \rho\cos\phi_2 \end{bmatrix}$$

We can make this a lower triangular matrix by switching, once again, the first and fourth columns of the Jacobian matrix; a factor of -1 is multiplied to the determinant of this matrix, which is nothing more than the product of the elements along its main diagonal:

$$\begin{aligned} J &= \csc\phi_2\rho\sec\phi_1\cos\phi_2\rho\sec\theta\cos\phi_1\cos\phi_2\rho\cos\theta\cos\phi_1\sin\phi_2 \\ &= \rho^3\cos\phi_1\cos^2\phi_2 \end{aligned}$$

Thus satisfying the sequence for $n = 4$. Therefore, by induction, the Jacobian of the n-sphere is given by

$$J = \rho^{n-1}\cos\phi_1\cos^2\phi_2\ldots\cos^{n-2}\phi_{n-2}$$

12) Let $S(r)$ be the n-sphere of radius r. Integrate the pullback of $\int_{S(r)} dx_1 dx_2 \dots dx_n$
and so prove that the volume of $S(r)$ is

$$\frac{2^{n/2}\, \pi^{n/2} r^n}{2\cdot 4\cdot 6 \cdots n} = \frac{\pi^{n/2}}{(n/2)!} r^n, \quad n \text{ is even}$$

$$\frac{2^{(n+1)/2}\, \pi^{(n-1)} r^2}{1\cdot 3\cdot 5 \cdot n} = \frac{[(n-1)/2]!}{n!} 2^n \pi^{(n-1)/2} r^n, \quad n \text{ odd}$$

Integral for 3-sphere

B = brick given by range of $\rho, \theta, \phi, \dots$,

$$\int_{S(r)} dx_1 dx_2 dx_3 \to \int_B \rho^2 \cos\phi_1 \, d\rho \, d\theta \, d\phi_1$$

*pullbacks

$$\int_0^r \rho^2 \, d\rho \to \int_0^r \frac{\rho^3}{3} \to \frac{r^3}{3}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^r \rho^2 \cos\phi_1 \, d\rho\, d\theta\, d\phi_1 = \left(\frac{r^3}{3}\right)(2\pi)(2) = \frac{4}{3}\pi r^3$$

$$\int_0^{2\pi} d\theta = 2\pi \qquad \int_{-\pi/2}^{\pi/2} \cos\phi_1 \, d\phi_1 = 2$$

$$\frac{[(2)/2]!}{3!}(2^3)(\pi^{2/2}) r^3 \to \frac{8}{6}\pi r^3 = \frac{4}{3}\pi r^3 \checkmark$$

this matches!

Integral for 4-sphere

$$\int_{S(r)} dx_1 dx_2 dx_3 dx_4 \to \int_B \rho^3 \cos\phi_1 \cos^2\phi_2 \, d\rho\, d\theta\, d\phi_1\, d\phi_2$$

$$\int_0^r \rho^3 = \frac{r^4}{4} \qquad \int_{-\pi/2}^{\pi/2} \cos^2\phi_2 \, d\phi_2 = \frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^r \rho^3 \cos\phi_1 \cos^2\phi_2 \, d\rho\, d\theta\, d\phi_1\, d\phi_2 = \left(\frac{r^4}{4}\right)(2\pi)(2)\left(\frac{\pi}{2}\right) = \frac{r^4\pi^2}{2}$$

even formula $= \frac{\pi^{4/2}}{(2!)} r^4 = \frac{r^4\pi^2}{2}$ $\checkmark$ checks

Integral for 5-sphere

$$\int_{S(r)} dx_1 dx_2 dx_3 dx_4 dx_5 \to \int_B \rho^4 \cos\phi_1 \cos^2\phi_2 \cos^3\phi_3 \, d\rho\, d\theta\, d\phi_1\, d\phi_2\, d\phi_3$$

$$\int_0^r \rho^4 = \frac{r^5}{5} \qquad \int_{-\pi/2}^{\pi/2} \cos^3\phi_3 = \frac{4}{3} \to \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^r \rho^4 \cos\phi_1 \cos^2\phi_2 \cos^3\phi_3 = \left(\frac{r^5}{5}\right)(2\pi)(2)\left(\frac{\pi}{2}\right)\left(\frac{4}{3}\right)$$

$$= \frac{16\pi^2 r^5}{30} = \frac{8}{15}\pi^2 r^5$$

check w/ formula: $2^5 \pi^{(4/2)} r^5 \left(\frac{2!}{5!}\right) = \frac{64}{120}\pi^2 r^5 = \frac{8}{15}\pi^2 r^5$ $\checkmark$

6-sphere

$$\int_{S(r)} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \to \int_B \rho^5 \cos\phi_1 \cos^2\phi_2 \cos^3\phi_3 \cos^4\phi_4 \, d\rho\, d\theta\, d\phi_1\, d\phi_2\, d\phi_3\, d\phi_4$$

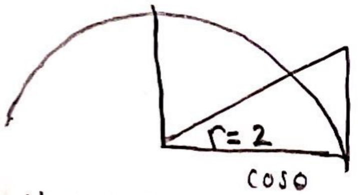
$$\int_0^r \rho^5 = \frac{r^6}{6} \qquad \int_{-\pi/2}^{\pi/2} \cos^4\phi_4 = \frac{3\pi}{8}$$

$$\to \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^r \rho^5 \cos\phi_1 \cos^2\phi_2 \cos^3\phi_3 \cos^4(\phi_4)\, d\rho\, d\theta\, d\phi_1\, d\phi_2\, d\phi_3\, d\phi_4 = \left(\frac{r^6}{6}\right)(2\pi)(2)\left(\frac{\pi}{2}\right)\left(\frac{4}{3}\right)\left(\frac{3\pi}{8}\right)$$

$$= \frac{48 r^6\pi^3}{288} = \frac{r^6\pi^3}{9}$$

check $= \frac{\pi^3 r^6}{3!} = \frac{r^6\pi^3}{9}$ $\checkmark$

So, there is a pattern to be noticed. To go to the next even/odd volume of an n-sphere, you multiply your old ($n-2$ sphere) volume by the integral of your new $\cos^{n-2}\phi_{n-2}\, d\phi_{n-2}$ and $\frac{r^n}{n}$, as the integral $\int_0^r \rho^{n-1}$ is trivial, and there's a clear pattern there. When multiplying the old $(n-2)$ volume by these new components we will get our desired n-volume, as the new integral of $\cos\phi_{n-2}\, d\phi_{n-2}$ accounts for our presence of π, and $\frac{r^n}{n}$ solves the changing of r.



polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

#5. Find the rate at which the flow

$$\vec{v} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, \frac{z}{x^2+y^2}\right)$$

crosses the cylindrical surface

$$S = \{(x,y,z) \mid x^2+y^2 = 4 \quad 0 \le z \le 2\}$$

oriented so that the positive direction is away from the z axis.

Using the parametrization:

$x = 2\cos\theta$ $\qquad 0 < \theta < 2\pi$

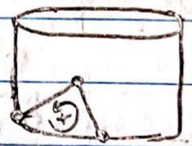
$y = 2\sin\theta$ $\qquad 0 \le z \le 2$

$z = z$

Let $\vec{F}(\theta, z) = (x, y, z)$ $\qquad \theta, z$, or z, θ ?

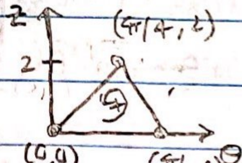
$$\left[\underset{A}{(2,0,0)}, \underset{B}{(0,2,0)}, \underset{C}{(\sqrt{2},\sqrt{2},2)}\right]$$

$$dF = \begin{array}{c|cc} & \theta & z \\ \hline x & -2\sin\theta & 0 \\ y & 2\cos\theta & 0 \\ z & 0 & 1 \end{array}$$



$\theta \quad z$

$[(0,0), (\pi/2, 0), (\pi/4, 2)]$



$F^*(dydz) = 2\cos\theta \, d\theta dz$

$F^*(dzdx) = 2\sin\theta \, d\theta dz$ $\qquad$ thus use $d\theta dz$

$F^*(dxdy) = 0$

$x^2+y^2 = (2\cos\theta)^2 + (2\sin\theta)^2 = 4$

$$\int_T v_1 dydz + v_2 dzdx + v_3 dxdy = \int_S \frac{x}{x^2+y^2} dydz + \frac{y}{x^2+y^2} dzdx + \frac{z}{x^2+y^2} dxdy$$

$$\int_S \left(\frac{\cos\theta}{2}(2\cos\theta) + \frac{\sin\theta}{2}(2\sin\theta) + \frac{z}{4}(0)\right) d\theta dz$$

$$= \int_S (\cos^2\theta + \sin^2\theta) d\theta dz = \int_0^2 \int_0^{2\pi} d\theta dz = \int_0^2 2\pi \, dz$$

$= 4\pi$

Let $\frac{\partial f}{\partial r} = f_r$ and $\frac{\partial f}{\partial \theta} = f_\theta$

11. $x = r\cos\theta$
$y = r\sin\theta$
$z = f(r,\theta)$

WMS that $d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\, r\,dr\,d\theta$

$$d\sigma = \left|\frac{\partial \vec{F}}{\partial r} \times \frac{\partial \vec{F}}{\partial \theta}\right| dr\,d\theta \text{ where } \vec{F} = (r\cos\theta, r\sin\theta, f(r,\theta))$$

$$d\sigma = \left|(-f_r r\cos\theta + f_\theta \sin\theta), (-f_\theta\cos\theta - f_r r\sin\theta), (r\cos^2\theta + r\sin^2\theta)\right| dr\,d\theta$$

$$= \sqrt{(f_\theta\sin\theta - f_r r\cos\theta)^2 + (-f_\theta\cos\theta - f_r r\sin\theta)^2 + r^2}\, dr\,d\theta$$

$$= \Big[f_r^2 r^2\cos^2\theta - \cancel{2f_r f_\theta r\cos\theta\sin\theta} + f_\theta \sin^2\theta + f_\theta^2\cos^2\theta$$
$$+ \cancel{2f_r f_\theta r\cos\theta\sin\theta} + f_r^2 r^2\sin^2\theta + r^2\Big]^{1/2} dr\,d\theta$$

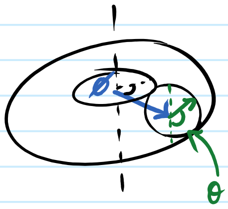
$$= \sqrt{f_r^2 r^2 + f_\theta^2 + r^2}\, dr\,d\theta$$

$$= \boxed{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\, dr\,d\theta}$$

S8.6 Exercise 13

Given: $x = (a+b\cos\theta)\sin\phi \qquad y = (a+b\cos\theta)\cos\phi \qquad z = b\sin\theta$

for a torus where $(a > b)$



$0 \le \phi, \theta \le 2\pi$

Parametrize:

$$\vec{F}(\theta,\phi) = \big((a+b\cos\theta)\sin\phi, (a+b\cos\theta)\cos\phi, b\sin\theta\big)$$

Surface area $S = \int_S d\sigma = \int_R \left|\frac{\partial \vec{F}}{\partial \theta} \times \frac{\partial \vec{F}}{\partial \phi}\right| d\theta\, d\phi$

$$\Rightarrow \frac{\partial \vec{F}}{\partial \theta} = (-b\sin\theta\sin\phi, -b\sin\theta\cos\phi, b\cos\theta)$$

$$\frac{\partial \vec{F}}{\partial \phi} = \big((a+b\cos\theta)\cos\phi, -(a+b\cos\theta)\sin\phi, 0\big)$$

$$\frac{\partial \vec{F}}{\partial \theta} \times \frac{\partial \vec{F}}{\partial \phi} = \begin{vmatrix} x & y & z \\ -b\sin\theta\sin\phi & -b\sin\theta\cos\phi & b\cos\theta \\ (a+b\cos\theta)\cos\phi & -(a+b\cos\theta)\sin\phi & 0 \end{vmatrix}$$

$$= \big((a+b\cos\theta)b\cos\theta\sin\phi, -(a+b\cos\theta)b\cos\theta\cos\phi, (a+b\cos\theta)b\sin\theta\sin^2\phi + (a+b\cos\theta)b\sin\theta\cos^2\phi\big)$$

$$= (a+b\cos\theta)b\,(\cos\theta\sin\phi, -\cos\theta\cos\phi, \sin\theta)$$

Magnitude of the cross-product $\left|\frac{\partial \vec{F}}{\partial \theta} \times \frac{\partial \vec{F}}{\partial \phi}\right|$:

$$(a+b\cos\theta)b\sqrt{\cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta} = b(a+b\cos\theta)$$

$$\int_S d\sigma = \int_R \left|\frac{\partial \vec{F}}{\partial \theta} \times \frac{\partial \vec{F}}{\partial \phi}\right| d\theta\, d\phi = b\int_0^{2\pi}\int_0^{2\pi} (a+b\cos\theta)\, d\theta\, d\phi$$

$$= b\int_0^{2\pi} 2\pi a + \cancelto{0}{b\sin 2\pi}\, d\phi = 4\pi ab$$

The surface area of the given torus is $4\pi ab$