Problem set 5 exemplars and comments.

You might have missed it, but somewhere along the way this course became geometric and visual. You cannot ignore that aspect. Calc III was heavily geometric and there are ways that our course is moreso. It is important to be able to reason both visually and also with equations. The first two problems on this problem set require visual reasoning. You need to not be averse to drawing pictures.

10.3.7 is about how paths fit together. Remember that \omega is not defined at the origin in this problem. You need to find a region not containing the origin that is contractible where it is defined. That is the basic idea - cutting the region into pieces that work nicely.

10.3.8 is for any two points. It is important to start with the two points and _then_ two paths connecting them, and finally that makes a curve to which the given can be applied.

10.3.14 is rather straightforward - differentiate omega and set the result equal to zero. Each term must be zero, so gather them together to get three equations. Reasons matter.

10.3.19 show closed by differentiating and then integrate by different variables. Do *NOT* add the results. Smoosh them together, or see how they are combined on pp. 292-293.

10.5.5 is rather straightforward - please get in the habit of using the FTC by differentiating, not by remembering ugly formulae. Remember changing order of integration is allowed as always in Calc III, it is not a sign change by changing orientation.

10.5.9 translate _first_ (as the directions say) so that there are no more triangles in your work. Then look at each side and see that they are related by the FTC (i.e. take the derivative of the 1-form and see that it equals the 2-form). Be careful _always_ each step on the way to match the dimensions of your integration location to the order of the form.

10.5.14 the torus here is produced by rotating the circle $(x-4)^2 + z^2 = 1$ around the z-axis.   The derivative of the 2-form is 3dxdydz, so this is, in the end, merely finding 3 times the volume of this particular torus. There is a parametrisation (including bounds) in the back. It is your job to find the Jacobian, either from a determinant of a matrix or by multiplying dx dy and dz in exterior algebra. The rho-theta-phi orientation works fine here. In this case because you are finding 3 times the volume, if you get a negative final answer, you know it is incorrect.

On both of these last problems (and always) you are required to follow directions. If it doesn't say how to do a problem, you are free to choose your own way. If it says "Use Gauß's theorem" and you don't, you have not done the problem.

Remember you have a quizam on this material coming soon (open 30 April - 1 May).

$\boxed{10.3}$ 7) let ω: closed 1-form in $\mathbb{R}^2$ w/ one singularity @ $(0,0)$.

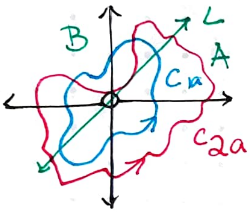
Show that if C_1 & C_2 are any pair of closed curves encircling (same direction) the origin, then
$$\oint_{C_1} \omega = \oint_{C_2} \omega$$

∘ note pg 293B "if no arrow is given, it is assumed that the [loop] curve is following the counterclockwise direction"

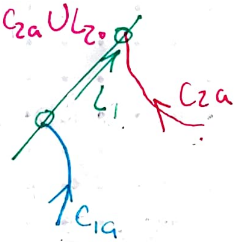
∘ Poincare's Lemma: if ω is closed on a contractible region, then ω is exact in that region

consider the line L crossing through $(0,0)$, dividing $\mathbb{R}^2$ into two contractible subsets A and B (one of the sets contain L). Then ω is exact over A and exact over B.

let C_{1a} & C_{2a} be the segments of loops C_1, C_2 in A, and let C_{1b}, C_{2b} be the segments of C_1, C_2 in B, and L_1, L_2 be the segments connecting C_{1a} and C_{2a}, etc.



now, consider the closed loops $C_{1a} \cup L_1 \cup -C_{2a} \cup L_2$.



Because ω is exact and $C_{1a} \cup L_1 \cup -C_{2a} \cup L_2$ is closed,
$$\int_{C_{1a} \cup L_1 \cup -C_{2a} \cup L_2} \omega = \int_{\emptyset} d\omega = 0$$

On the other side, we have an opposite oriented loop (C_{1a} is oriented ccwise)
$$\int_{C_{1b} \cup -L_2 \cup C_{2b} \cup -L_1} \omega = \int_{\emptyset} d\omega = 0$$

composing our two loops cancels the contribution along L_1 and L_2

$$\int_{C_{1a}} \omega + \int_{L_1} \omega + \int_{-C_{2a}} \omega + \int_{L_2} \omega \quad \int_{C_{1b}} \omega + \int_{-L_2} \omega + \int_{-C_{2b}} \omega + \int_{-L_1} \omega = 0$$

Grouping like terms:
$$\left(\int_{C_{1a}} \omega + \int_{C_{1b}} \omega\right) + \left(\int_{C_{2a}} \omega + \int_{C_{2b}} \omega\right) + \left(\int_{L_1} \omega + \int_{-L_1} \omega\right) + \left(\int_{L_2} \omega + \int_{-L_2} \omega\right) = 0$$

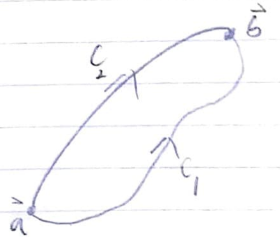
$$\oint_{C_1} \omega \quad + \quad \int_{-C_2} \omega \quad + \quad 0 \quad + \quad 0 \quad = 0$$

$$\oint_{C_1} \omega - \oint_{C_2} \omega = 0 \quad \square$$

8 show that if w is a differential 1-form in R^2 for which the integral around any closed curve is 0 then the integral of w between any two points is independant of the path

we know any integral w over a closed curve

$$\oint_C w \, dw = 0$$



now assumme we have two points a and b that are on a closed path C

let C_1 and C_2 be two path inside R that both start at a and end at b

let C be the curve obtained by taking the C_1 path from $\vec{a}$ to $\vec{b}$ and the going from $\vec{b}$ to $\vec{a}$ along C_2 in the reverse direction

Since the curve C is closed that means

$$0 = \oint_C w = \int_{C_1} w \int_{C_2} w \qquad \int_{C_1} w = \int_{C_2} w$$

therefore the integral of w between any two point is independent of the path

(14) For a closed 1-form of the form

$\omega = f\,dx + g\,dy + h\,dz$, find the partial differential equation that must hold for $d\omega = 0$ to be true.

$$\omega = f\,dx + g\,dy + h\,dz$$

$$d\omega = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)dx$$

$$+\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right)dy$$

$$+\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz\right)dz = 0$$

$$= \frac{\partial f}{\partial y}dy\,dx + \frac{\partial f}{\partial z}dz\,dx + \frac{\partial g}{\partial x}dx\,dy + \frac{\partial g}{\partial z}dz\,dy$$

$$+\frac{\partial h}{\partial x}dx\,dz + \frac{\partial h}{\partial y}dy\,dz$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)dz\,dx$$

$$+\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy\,dz = 0$$

$\therefore$ the following PDEs hold:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x},\ \frac{\partial h}{\partial y} = \frac{\partial g}{\partial z}$$

Problem 10.3.19:

1. Let $\omega = (2x+y)dx + (x+zt)dy + (yt-t)dz + (yz-z)dt$. For a differential form to be closed, we must satisfy $d\omega = 0$.
   Taking the differential of the 1-form,

$$\begin{aligned} d\omega &= d((2x+y)dx + (x+zt)dy + (yt-t)dz + (yz-z)dt) \\ &= (2dx+dy)dx + (dx+tdz+zdt)dy + (tdy+ydt-dt)dz + (zdy+ydz-dz)dt \\ &= dydx + dxdy + t\,dzdy + z\,dtdy + t\,dydz + y\,dtdz - dtdz + z\,dydt + y\,dzdt - dzdt \\ &= 0 \end{aligned}$$

   Therefore, ω is closed and as a result, a scalar field F exists that satisfies $\omega = dF$.

Finding dF calls for integrating each individual "component" of the 1-form ω:

$$\frac{\partial F}{\partial x} = 2x + y \longrightarrow \int \frac{\partial F}{\partial x} dx = \int 2x + y\, dx$$
$$\implies F(x, y, z, t) = x^2 + xy + f(y, z, t)$$
$$\frac{\partial F}{\partial y} = x + zt \longrightarrow \int \frac{\partial F}{\partial y} dy = \int x + zt\, dy$$
$$\implies F(x, y, z, t) = xy + yzt + g(x, z, t)$$
$$\frac{\partial F}{\partial z} = yt - t \longrightarrow \int \frac{\partial F}{\partial z} dz = \int yt - t\, dz$$
$$\implies F(x, y, z, t) = yzt - zt + h(x, y, t)$$
$$\frac{\partial F}{\partial t} = yz - z \longrightarrow \int \frac{\partial F}{\partial t} dt = \int yz - z\, dt$$
$$\implies F(x, y, z, t) = yzt - zt + j(x, y, z)$$

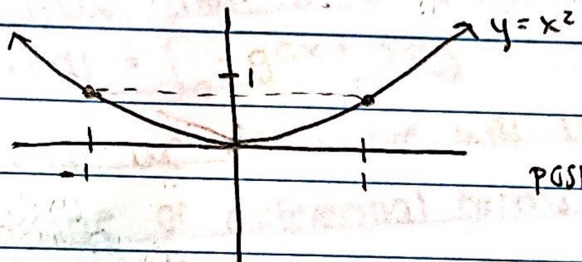
Thus by inspection, the scalar field should take the form

$$F(x, y, z, t) = x^2 + xy + yzt - zt$$

#5. Use Green's theorem to evaluate

$$\oint_C y\,dx + x^2\,dy$$

where C follows the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$ and then returns on the straight line from $(1,1)$ to $(-1,1)$



$-1 \le x \le 1$

$x^2 \le y \le 1$

positive orientation

Green's Theorem is $\int_C P\,dx + Q\,dy = + \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$\oint_C y\,dx + x^2\,dy = \int_{-1}^{1}\int_{x^2}^{1} 2x - 1 \; dy\,dx = \int_{-1}^{1} \Big[2xy - y\Big]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \Big[(2x-1) - (2x^3 - x^2)\Big] dx = \int_{-1}^{1} 2x - 1 - 2x^3 + x^2 \; dx$$

$$= \left[\frac{2x^2}{2} - x - \frac{2x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \left(\cancel{1-1} - \frac{1}{2} + \frac{1}{3}\right) - \left(1 + 1 - \frac{1}{2} - \frac{1}{3}\right) = -4/3$$

9) a) $\oint_{\partial R} \frac{\partial g}{\partial n} ds = \oint_{\partial R} \nabla g \cdot \vec{n}\, ds = \oint_{\partial R} -\frac{\partial g}{\partial y} dx + \frac{\partial g}{\partial x} dy$

$\int_R \nabla^2 g\, dx dy = \int_R \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dx dy$

By the fundamental theorem of calculus, we know that $\oint_{\partial M} \omega = \int_M d\omega$ If we let $\omega = \frac{-\partial g}{\partial y} dx + \frac{\partial g}{\partial x} dy$, then $d\omega = d\left(-\frac{\partial g}{\partial y} dx + \frac{\partial g}{\partial x} dy\right) = d\left(-\frac{\partial g}{\partial y}\right) dx + d\left(\frac{\partial g}{\partial x}\right) dy = \left(dx - \frac{\partial^2 g}{\partial y^2} dy\right) dx + \left(\frac{\partial^2 g}{\partial x^2} dx + dy\right) dy = -\frac{\partial^2 g}{\partial y^2} dy dx + \frac{\partial^2 g}{\partial^2 x} dx dy = \left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial x^2}\right) dx dy$ Thus,

$$\oint_{\partial R} \frac{\partial g}{\partial n} ds = \oint_{\partial R} -\frac{\partial g}{\partial y} dx + \frac{\partial g}{\partial x} dy = \int_R \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dx dy = \int_R \nabla^2 g\, dx dy$$

b) $\oint_{\partial R} f \frac{\partial g}{\partial n} ds = \oint_{\partial R} f \nabla g \cdot \vec{n}\, ds = \oint_{\partial R} -f \frac{\partial g}{\partial y} dx + f \frac{\partial g}{\partial x} dy$

$\int_R (f \nabla^2 g + \nabla f \cdot \nabla g) dx dy = \int_R \left(f \frac{\partial^2 g}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y}\right) dx dy$

If we let $w = -f \frac{\partial g}{\partial y} dx + f \frac{\partial g}{\partial x} dy$, then $dw = d\left(-f \frac{\partial g}{\partial y}\right) dx + d\left(f \frac{\partial g}{\partial x}\right) dy$
$= \left(-f\left(dx + \frac{\partial^2 g}{\partial y^2} dy\right) + \left(\frac{-\partial f}{\partial x} dx + \frac{-\partial f}{\partial y} dy\right) \frac{\partial g}{\partial y}\right) dx + \left(f\left(\frac{\partial^2 g}{\partial x^2} dx + dy\right) + \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \cdot \frac{\partial g}{\partial x}\right) dy$
$= \left(f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial x}\right) dx dy$. Thus, by the fundamental theorem of calculus

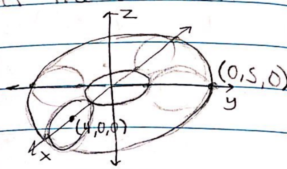
$$\oint_{\partial R} f \frac{\partial g}{\partial n} ds = \oint_{\partial R} -f \frac{\partial g}{\partial y} dx + f \frac{\partial g}{\partial x} dy = \int_R \left(f \frac{\partial^2 g}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y}\right) dx dy$$

$= \int_R (f \nabla^2 g + \nabla f \cdot \nabla g) dx dy$, as desired use parametrization $x=(4+\rho\cos\phi)\cos\theta$

$0\le\rho\le 1$ $\quad y=(4+\rho\cos\phi)\sin\theta$

$0\le\theta\le 2\pi \quad 0\le\phi\le 2\pi \quad z=\rho\sin\phi$

14) Sketch the torus given by the equation $(r-4)^2+z^2=1$, where $x=r\cos\theta$ and $y=r\sin\theta$. Use Gauss's Theorem to determine the rate @ which the flow described by $x\,dydz+y\,dzdx+z\,dxdy$ crosses the surface of this torus.



$$\begin{array}{c|ccc} & \rho & \theta & \phi \\ x & \cos\phi\cos\theta\,d\rho & -(4+\rho\cos\phi)\sin\theta\,d\theta & -\rho\sin\phi\cos\theta\,d\phi \\ y & \cos\phi\sin\theta\,d\rho & (4+\rho\cos\phi)\cos\theta\,d\theta & -\rho\sin\phi\sin\theta\,d\phi \\ z & \sin\phi\,d\rho & 0\,d\theta & \rho\cos\phi\,d\phi \end{array}$$

$dx=\cos\phi\cos\theta\,d\rho-(4+\rho\cos\phi)\sin\theta\,d\theta-\rho\sin\phi\cos\theta\,d\phi$

$dy=\cos\phi\sin\theta\,d\rho+(4+\rho\cos\phi)\cos\theta\,d\theta-\rho\sin\phi\sin\theta\,d\phi$

$dz=\sin\phi\,d\rho + 0\,d\theta+\rho\cos\phi\,d\phi$

$$w=x\,dydz+y\,dzdx+z\,dxdy$$

$$dw=((1)+(1)+(1))\,dxdydz=3\,dxdydz$$

$dxdy=\cos\phi\cos\theta\,d\rho(4+\rho\cos\phi)\cos\theta\,d\theta-\cos\phi\cos\theta\rho\sin\phi\sin\theta\,d\rho d\phi-(4+\rho\cos\phi)\sin^2\theta\cos\phi\,d\theta d\rho$

$\quad+\rho\sin\phi\sin^2\theta(4+\rho\cos\phi)d\theta d\phi-\rho\sin\phi\cos\theta\cos\phi\sin\theta\,d\phi d\rho-(4+\rho\cos\phi)\cos^2\theta\rho\sin\phi\,d\phi d\theta$

$\quad=\cos\phi\cos^2\theta(4+\rho\cos\phi)d\rho d\theta+(4+\rho\cos\phi)\sin^2\theta\cos\phi\,d\rho d\theta-\cancel{\rho\cos\phi\cos\theta\sin\phi\sin\theta\,d\rho d\phi}$

$\quad+\cancel{\rho\sin\phi\cos\theta\cos\phi\sin\theta\,d\rho d\phi}+\rho\sin\phi\sin^2\theta(4+\rho\cos\phi)d\theta d\phi+(4+\rho\cos\phi)\cos^2\theta\rho\sin\phi\,d\theta d\phi$

$dxdy=\cos\phi(4+\rho\cos\phi)d\rho d\theta+\rho\sin\phi(4+\rho\cos\phi)d\theta d\phi$

$dxdydz=\sin\phi(\rho\sin\phi)(4+\rho\cos\phi)d\rho d\theta d\phi+\rho\cos\phi\cos\phi(4+\rho\cos\phi)d\phi d\rho d\theta$

$dxdydz=\sin^2\phi\,\rho(4+\rho\cos\phi)d\rho d\theta d\phi+\cos^2\phi\,\rho(4+\rho\cos\phi)d\rho d\theta d\phi$

$dxdydz=\rho(4+\rho\cos\phi)d\rho d\theta d\phi$

$$\iiint 3\,dxdydz=\int_0^{2\pi}\int_0^{2\pi}\int_0^1(12\rho+3\rho\cos\phi)\,d\rho d\theta d\phi$$

$$=\int_0^{2\pi}\int_0^{2\pi}(12+3\cos(\phi))\,d\theta d\phi$$

$$=\int_0^{2\pi}\pi(12+3\cos(\phi))\,d\phi$$

$$=12\pi\phi+3\pi\sin(\phi)\Big|_0^{2\pi}=\boxed{24\pi^2}$$

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