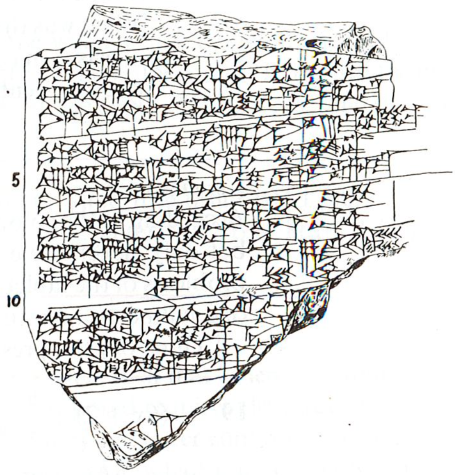
1.E1 Some Babylonian problem texts

(a) YBC 4652



Transcription

19 ^{1}na$_4$ ʾ-pá ki-lá nu-na-tag 6-bi ì-lá 2 gín [bí-daḫ-ma]
^{2}igi-3-gál igi-7-gál a-rá-24-kam tab bí-daḫ-ma
^{3}ì-lá 1 ma-na sag na$_4$ en-nam sag na$_4$ $4\frac{1}{3}$ gín

20 ^{4}na$_4$ ì-pà ki-lá nu-na-tag 8-bi ì-lá 3 gín bí-daḫ-ma
^{5}igi-3-gál igi-13-gál a-rá 21 e-tab bí-daḫ-ma
^{6}ì-lá 1 ma-na sag na$_4$ en-nam sag na$_4$ $4\frac{1}{2}$ gín

21 ^{7}na$_4$ ì-pà ki-lá nu-na-tag igi-6-gál ba-zi
^{8}igi-3-gál igi-8-gál bí-daḫ-ma ì-lá 1 ma-na
^{9}sag na$_4$ en-nam sag na$_4$ 1 ma-na 9 gín $21\frac{1}{2}$ še
^{10}ù ⟨igi-⟩ 10-gál še kam

Translation

19 [1]I found a stone, (but) did not weigh it; (after) I weighed (out) 6 times (its weight), [added] 2 gín, (and)
[2]added one-third of one-seventh multiplied by 24,
[3]I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was $4\frac{1}{3}$ gín.

20 [4]I found a stone, (but) did not weigh it; (after) I weighed (out) 8 times (its weight), added 3 gín,
[5]one-third of one-thirteenth I multiplied by 21, added (it), and then
[6]I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was $4\frac{1}{2}$ gín.

21 [7]I found a stone, (but) did not weigh it; (after) I subtracted one-sixth (and)
[8]added one-third of one-eighth, I weighed (it): 1 ma-na.
[9]What was the origin(al weight) of the stone? The origin(al weight) of the stone was 1 ma-na, 9 gín, $21\frac{1}{2}$ še,
[10]and one-tenth of a še.

Commentary
According to the colophon, the tablet contained twenty-two examples when complete, but only eleven are even partly preserved; of these, six can be fully restored. All the problems of the tablet are obviously of the same type, resulting in a linear equation for one unknown quantity, the 'original' weight of a stone.

[Three of] the preserved problems are the following:

19 $$(6x+2)+\frac{1}{3}\cdot\frac{1}{7}\cdot 24(6x+2)=1,0$$

Solution: $x = 4\frac{1}{3}$ gín.

Indeed, $6x+2=28 \qquad \frac{24}{21}(6x+2)=32.$

20 $$(8x+3)+\frac{1}{3}\cdot\frac{1}{13}\cdot 21(8x+3)=1,0$$

Solution: $x = 4\frac{1}{2}$ gín.

Indeed, $8x+3=39 \qquad \frac{21}{39}(8x+3)=21.$

21 $$\left(x-\frac{x}{6}\right)+\frac{1}{3}\cdot\frac{1}{8}\left(x-\frac{x}{6}\right)=1,0$$

Solution: $x = 1$ ma-na $+ 9$ gín $+ 21\frac{1}{2}$ še $+ \frac{1}{10}$ še $= 1,9;7,12$ gín.

Indeed, $\frac{x}{6}=11;31,12 \qquad x-\frac{x}{6}=57;36 \qquad \frac{1}{24}\left(x-\frac{x}{6}\right)=2;24.$

The remaining problems are completely broken away or too badly preserved to be restored with certainty.

The main difficulty encountered in interpreting the text of the problem consists in placing the parentheses correctly. The terminology alone is in itself inadequate; only experience with analogous problems, when combined with the given solution, indicates the correct interpretation. The ancient scribes of course had the oral interpretation of their teachers at their disposal.

(*b*) *YBC 4186*

[1]A cistern was 10 GAR square, 10 GAR deep.
[2-3]I emptied out(?) its water; with its water how much field did I irrigate to a depth of 1 šu-si?
[4]Put (aside) 10 and 10 which formed the square.
[5]Put (aside) 10, the depth of the cistern.
[6-6a]And put (aside) 0; 0, 10, the depth of the water which irrigated the field.
[7-8]Take the reciprocal of 0; 0, 10, the depth of the water which [irri]gated the field, and (the resulting) 6, 0 [mul]tiply by 10, the depth of the cistern, (and the result is) 1, 0, 0.
[9]1, 0, 0 ke[ep] in your head.
[10][Square (?)] 10, which formed the square, [and (the result is)] 1, 40.
[11-12]Multiply 1, 40 by 1, 0, 0, which you are ke[eping] in your head. I irrigated 1, 40, 0, 0 (SAR) field.

Commentary
The text assumes a cistern (túl) in the shape of a cube, such that its length l, width b, and depth h are 10 GAR each. The problem which is posed requires the calculation of the area A of a field irrigated to a depth h_A of 1 šu-si by the water contained in the cistern. After the transformation of $h_A = 1$ šu-si to 0; 0, 10 GAR, which is necessary because h is expressed in units of GAR, is made, the actual computation is carried out according to the formula

$$\frac{h}{h_A}\cdot l\cdot b = A.$$

The transformation of the final answer 1, 40, 0, 0 (SAR) to the standard 3 šár 2 bur'u is not made in the text.

The situation described in the text is strongly idealized in that the water is required to be spread to a uniform depth of one finger's breadth over a field which is approximately $3\frac{1}{2}$ kilometers square.

(*c*) *YBC 4608*

[13]A triangle. 6, 30 is the length, [11, 22], 30 the area; I did not know [its (?)] width.
[14]6 brothers divided it. One brother('s share) exceeded the other('s), but

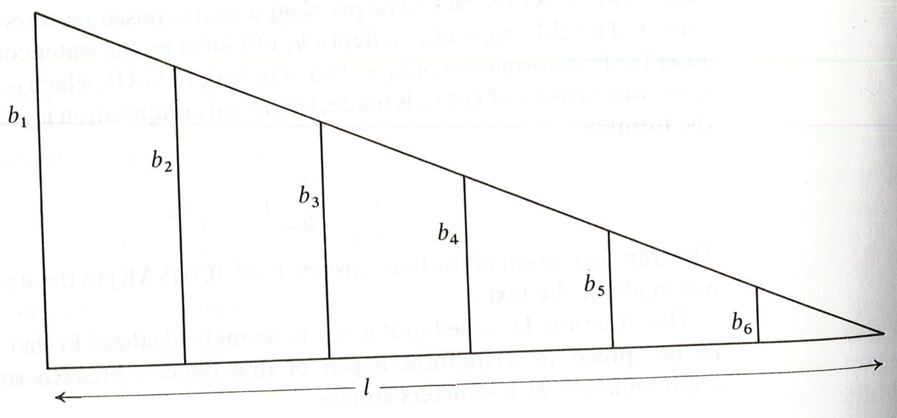
[15]how much he exceeded I did not know.
[16]How much did one brother exceed the other?
[17]When you perform (the operations), multiply the area by two, (and the result is) 22, 45, 0.
[18-19]The reciprocal of 6, 30 is not obtainable. What should I put to 6, 30 which will give me 22, 45, 0? Put 3, 30, (which is)
[20]the upper width. Take the reciprocal of 6, the brothers, (and) [multiply] the (resulting) 0; 10 by 6, 30, and (the resulting)
[21]1, 5 (is) the length which each too[k ......]
[22]35 GAR is the breadth. 35 [from 3, 30 ......]
[23]35 from 2, 5[5 ......]
[24]35 from 2, 2[0 ......]
[25]35 from 1, 4[5 ......]
[26]subtr[act] 35 from 1, 10 [......]
[27]subtract 35, and the width (?) [......]

Commentary

We have here one of the 'inheritance' problems which require the partition of property to be distributed among a given number of brothers. The field in question is of triangular shape with length l and area A:

$$l = 6, 30 \qquad A = 11, 22, 30.$$

This area is divided among 6 brothers by equidistant lines parallel to the base of the triangle. The question asked by the text concerns the difference between the allotments of the brothers.



(*d*) *YBC 6967*

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[1][The *igib*]*ūm* exceeded the *igūm* by 7.
[2]What are [the *igūm* and] the *igibūm*?

[3-5]As for you—halve 7, by which the *igibūm* exceeded the *igūm*, and (the result is) 3; 30.
[6-7]Multiply together 3; 30 with 3; 30, and (the result is) 12; 15.
[8]To 12; 15, which resulted for you,
[9]add [1, 0, the produ]ct, and (the result is) 1, 12; 15.
[10]What is [the square root of 1], 12; 15? (Answer:) 8; 30.
[11]Lay down [8; 30 and] 8; 30, its equal, and then
(*Reverse*)
[1-2]subtract 3; 30, the *takīltum*, from the one,
[3]add (it) to the other.
[4]One is 12, the other 5.
[5]12 is the *igibūm*, 5 the *igūm*.

Commentary

The problem treated here belongs to a well-known class of quadratic equations characterized by the terms igi and igi-bi (in Akkadian, *igūm* and *igibūm*, respectively). These terms refer to a pair of numbers which stand in the relation to one another of a number and its reciprocal, to be understood in the most general sense as numbers whose product is a power of 60. We must here assume the product

$$xy = 1, 0 \qquad (1)$$

as the first condition to which the unknowns x and y are subject. The second condition is explicitly given as

$$x - y = 7. \qquad (2)$$

From these two equations it follows that x and y can be found from

$$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \sqrt{\left(\frac{7}{2}\right)^2 + 1, 0} \pm \frac{7}{2},$$

a formula which is followed exactly by the text, leading to

$$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \sqrt{1, 12; 15} \pm 3; 30 = 8; 30 \pm 3; 30 = \left\{\begin{matrix} 12 \\ 5 \end{matrix}\right.$$

(*e*) *YBC 4662*

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21 [24]A ki-lá. $7\frac{1}{2}$ SAR is the area, 45 SAR the volume; one-seventh
[25]of that by which the length exceeded the width is its depth. What are the length, the width, and its depth?
[26]When you perform (the operations), take the reciprocal of $7\frac{1}{2}$ SAR, the area, [multiply by] 45, [the volume, (and)]
[27]you will get its depth. Halve the one-seventh which has been assumed, (and)
[28]you will get 3; 30. Take the reciprocal of its depth, (and) you will get 0; 10;
[29]multiply 0; 10 by 45 (SAR), the volume, (and) you will get 7; 30.
[30-31]Halve 3; 30, (and) you will get 1; 45; multiply together 1; 45 times 1; 45, (and) you will get 3; 3, 45; add 7; 30 to 3; 3, 45, (and)

[32]you will get 10; 33, 45; as for 10; 33, 45, [take] its square root, (and)
[33]you will get 3; 15; operate with 3; 15 ⟨twice⟩:
[34]add 1; 45 to one, subtract 1; 45 from the other, (and)
[35]you will get the length and the width. 5 GAR is the length; [$1\frac{1}{2}$ GAR is the width].

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22 [36]A ki[-lá. 5 GAR is the length, $1\frac{1}{2}$ GAR the width], $\frac{1}{2}$ GAR its depth, 10 [gín (volume) the assignment].
[37][How much length did one man take? When] you perform (the operations),
[38][multiply together the width and its depth, (and) you] will get 9;
[39-40][take the reciprocal of 9, (and)] you will get [0; 6, 40; multiply] 0; 6, 40 times the assignment, (and) you will get [0; 1, 6, 40]. 0; 1, 6, 40 (GAR) is the taking of one man.

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23 [41][A ki-lá. 5 GAR is the length, $1\frac{1}{2}$ GAR the width, $\frac{1}{2}$ GAR] its depth, 10 gín (volume) the assignment.
[42][How much length did 30 workers take?] When you [perform (the operations)],
...(three or four

(*Reverse*)

lines missing)...

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24 [3][A ki-lá. 5 GAR is the length, $1\frac{1}{2}$ GAR the width, $\frac{1}{2}$ GAR] its [depth], 10 gín (volume) [the assignment].
[4]In how many [days] did [30 workers] finish?
[5]When you perform (the operations), multiply together the length and the width, (and)
[6]you will get [7;] 30; multiply 7; 30 by its depth, (and) you will get 45.
[7]Take the reciprocal of the assignment, (and) you will get 6; multiply 45 by 6, (and)
[8]you will get 4, 30. Take the reciprocal of 30 workers, (and) you will get 0; 2;
[9]multiply by 4, 30, (and) you will get 9.
[10]30 workers finished on the 9th day.

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25 [11]A ki-lá. $1\frac{1}{2}$ GAR is the width, $\frac{1}{2}$ GAR its depth, 10 gín (volume) the assignment;
[12]30 workers finished on the 9th day.
[13]What is its length? When you perform (the operations),
[14]multiply together the width and its depth, (and) you will get 9. Take the reciprocal of the assignment, [(and) you will get 6];
[15]multiply 6 by 9, (and) you will get 54; take the reciprocal of 54, (and) you will get 0; 1, 6, 40.
[16-17]Multiply together 30 and 9, (and) you will get 4, 30; multiply 4, 30 by 0; 1, 6, 40, (and) you will get the length. 5 GAR is the length.

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26 [18]A ki-lá. 5 GAR is the length, $\frac{1}{2}$ GAR its depth, 10 gín (volume) the assignment; 30 workers
[19]finished on the 9th day. What is its width?

[20]When you perform (the operations), multiply together the length and its depth, (and)
[21]you will get [3]0. Take the reciprocal of the assignment, (and) you will get 6;
[22]multiply [30] by 6, (and) ⟨you will get 3, 0⟩; take ⟨the reciprocal⟩ of 3, 0, (and) you will see 0; 0, 20. 30 workers and 9
[23][multiply] together, (and) you will get 4, 30; multiply 4, 30 by 0; 0, 20, and
[24]you will get the width. $1\frac{1}{2}$ GAR is the width.

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27 [25]A ki[-lá. 5 GAR is the length, $1\frac{1}{2}$ GAR] the width, 10 gín (volume) the assignment; ⟨30 workers finished on the 9th day⟩.
[26]What is its depth? When [you] perform (the operations),
[27]multiply together the length and the width, (and) you will get [7; 30]. Take the reciprocal of the assignment, ⟨multiply by 7; 30⟩, (and)
[28]you will get 45; take the reciprocal of 45, (and) you will get 0; 1, 20.
[29]Multiply [together] 30 workers (and) the 9th day, (and) you will get [4, 3]0;
[30]multiply 4, 30 by 0; 1, 20, [(and) you will get 6. $\frac{1}{2}$ GAR is its depth].

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28 [31]A ki-lá. 5 GAR is the length, [$1\frac{1}{2}$ GAR the width], $\frac{1}{2}$ GAR its depth; ⟨30 workers finished on the 9th day.⟩ What is the assignment?
[32]When you perform (the operations), multiply together the length and the width, (and)
[33]you will see 7; 30; multiply 7; 30 by its depth, (and) you will see 45.
[34]Multiply together 30 workers and the 9th day, (and) you will see 4, 30;
[35]take the reciprocal of 4, 30, (and) you will see 0; 0, 13, 20; [multiply] 0; 0, 13, 20 by [45], (and)
[36]you will get the assignment. 10 gín (volume) is the as[signment].

(*f*) *BM 13901*

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1 I have added up the area and the side of my square: 0; 45. You write down 1, the coefficient. You break off half of 1. 0; 30 and 0; 30 you multiply: 0; 15. You add 0; 15 to 0; 45: 1. This is the square of 1. From 1 you subtract 0; 30, which you multiplied. 0; 30 is the side of the square.

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2 I have subtracted the side of my square from the area: 14, 30. You write down 1, the coefficient. You break off half of 1. 0; 30 and 0; 30 you multiply. You add 0; 15 to 14, 30. Result 14, 30; 15. This is the square of 29; 30. You add 0; 30, which you multiplied, to 29; 30. Result 30, the side of the square.

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7 I have added up seven times the side of my square and eleven times the area: 6; 15. You write down 7 and 11. You multiply 6; 15 by 11: 1, 8; 45. You break off half of 7. 3; 30 and 3; 30 you multiply. 12; 15 you add to 1, 8; 45. Result 1, 21. This is the square of 9. You subtract 3; 30, which you multiplied, from 9. Result 5; 30. The reciprocal of 11 cannot be found. By what must I multiply 11 to obtain 5; 30? 0; 30, the side of the square is 0; 30.

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10 The surfaces of my two square figures I have taken together: 21; 15. The side of one is a seventh less than the other. You write down 7 and 6. 7 and 7 you multiply: 49. 6 and 6 you multiply. 36 and 49 you add: 1; 25. The reciprocal of 1; 25 cannot be found. By what must I multiply 1; 25 to give me 21; 15? 0; 15. 0; 30 the side. 0; 30 to 7 you raise: 3; 30 the first side. 0; 30 to 6 you raise: 3 the second side.

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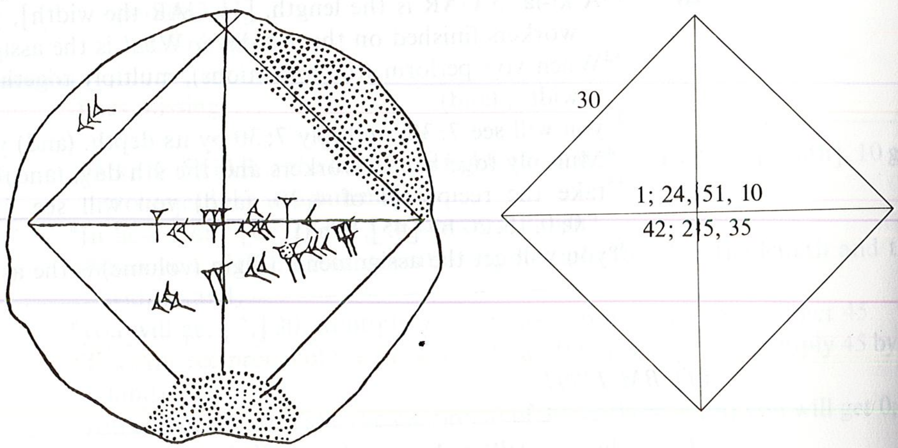
12 The surfaces of my two square figures I have taken together: 21; 40. The sides of my two square figures I have multiplied: 10, 0. You break off half of 21; 40. 10; 50 and 10; 50 you multiply: 1, 57, 21, 40. 10, 0 and 10, 0 you multiply, 1, 40, 0, 0 inside 1, 57, 21, 40 you tear off: 17, 21, 40. 4, 10 the side. 4, 10 to the first 10, 50 you add: 15, 0. 30 the side. 30 the first square figure. 4, 10 inside the second 10, 50 you tear off: 6, 40. 20 the side. 20 the second square figure.

(*g*) *YBC 7289*



The number 30 indicates the side a of the square, and 1, 24, 51, 10 means

(1) $$1;24,51,10 \approx \sqrt{2};$$

we therefore find

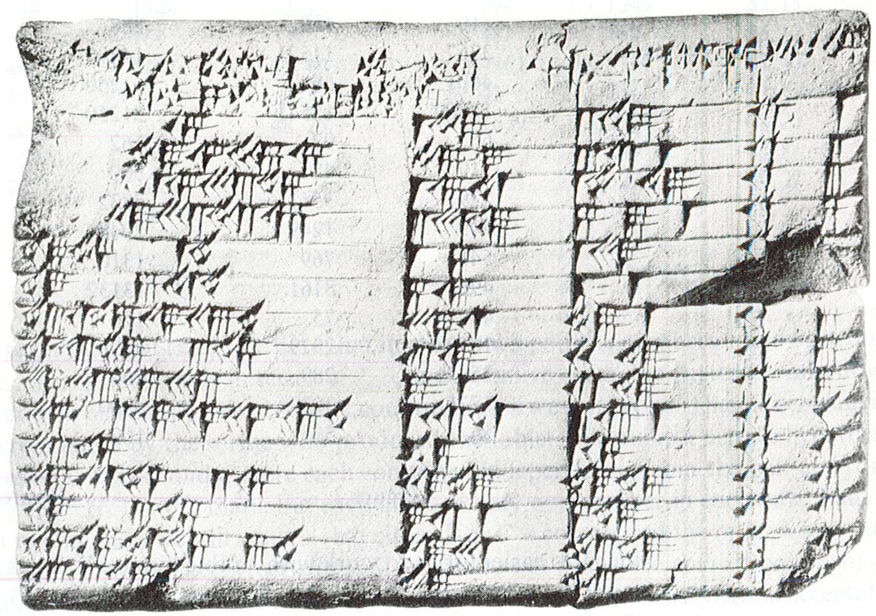
$$d = a\sqrt{2} = 42;25,35$$

for the diagonal. The value (1) for $\sqrt{2}$ is very good, as can be seen from

$$(1;24,51,10)^2 = 1;59,59,59,38,1,40.$$

1.E2 Sherlock Holmes in Babylon: an investigation by R. Creighton Buck

With this brief introduction to the arithmetic of the Babylonians, we turn to another tablet whose mathematical nature had been overlooked until the work of Neugebauer and Sachs [see 1.E1]. It is in the George A. Plimpton Collection, Rare Book and Manuscript Library, at Columbia University, and usually called Plimpton 322. The left side of this tablet has some erosion: traces of modern glue on the left edge suggest that a portion that had originally been attached there has since been lost or stolen. Since it was bought in a marketplace, one may only conjecture about its true origin and date, although the style suggests about -1600 for the latter. As with most such tablets, this had been assumed to be a commercial account or inventory report. We will attempt to show why one can be led to believe otherwise.



Column *A*	Column *B*	Column *C*
15	1/59	2/49
58/14/50/6/15	56/7	3/12/1
1/15/33/45	1/16/41	1/50/49
5 29/32/52/16	3/31/49	5/9/1
48/54/ 1/40	1/5	1/37
47/ 6/41/40	5/19	8/1
43/11/56/28/26/40	38/11	59/1
41/33/59/ 3/45	13/19	20/49
38/33/36/36	9/1	12/49
35/10/2/28/27/24/26/40	1/22/41	2/16/1
33/45	45	1/15
29/24/54/ 2/15	27/9	48/49
27/ 3/45	7/12/1	4/49
25/48/51/35/6/40	29/31	53/49
23/13/46/40	56	53

Figure 1

First, let us transcribe it into the slash notation, as seen in Figure 1. We have reproduced the three main columns, which we have labelled A, B, and C. We note that there are gaps in column A, due to the erosion. However, it seems apparent that the numbers there are steadily decreasing. We note that some of the numerals there are short and some long, apparently at random. In contrast with this, all the numerals in columns B and C are rather short, and we do not see any evidence of general monotonicity.

B	C
119	169
3367	11521
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
541	769
4961	8161
45	75
1679	2929
25921	289
1771	3229
56	53

Figure 2

$C+B$	$C-B$
288	50
14888	8154
11250	2048
31250	5832
162	32
800	162
5832	1250
2048	450
1310	228
13132	3200
120	30
4608	1250
26210	−25632
5000	1458
109	−3

Figure 3

Since it is easier for us to work with Arabic numerals, let us translate columns B and C into these numerals and look for patterns. (See Figure 2.) We see at once that B is smaller than C, with only two exceptions. Also, playing with these numbers, we find that column B contains exactly one prime, namely, 541, while column C contains eight numbers that are prime.

In the first 20,000 integers, there are about 2,300 primes, which is about 10 per cent; among 15 integers, selected at random from this interval, we might, then, expect to see one or two primes, but certainly not eight! This at once tells us that the tablet is mathematical and not merely arithmetical. (Imagine your feelings if you were to find a Babylonian tablet with a list of the orders of the first few sporadic simple groups.)

Encouraged, one attempts to find further visible patterns, for example, by combining the entries in columns B and C in various ways. One of the earliest tries is immediately successful. In Figure 3, we show the results of calculating $C+B$ and $C-B$. If you are sensitive to arithmetic you will note that, in almost every case, the numbers are each twice a perfect square.

If $C+B=2a^2$ and $C-B=2b^2$, then $B=a^2-b^2$ and $C=a^2+b^2$. Thus the entries in these columns could have been generated from integer pairs (a,b). In passing, we note that B, being $(a-b)(a+b)$, is not apt to be prime; on the other hand, when a and b are relatively prime, every prime of the form $4N+1$ can be expressed as a^2+b^2.

In Figure 4, we have recopied columns B and C, together with the appropriate pairs (a,b) in the cases where this representation is possible. As a further confirmation that

B	C	(a,b)
119	169	12, 5
3367	11521	?
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
541	769	?
4961	8161	81, 40
45	75	?
1679	2929	48, 25
25921	289	?
1771	3229	50, 27
56	53	?

Figure 4

Corrected Version

B	C	(a,b)
119	169	12, 5
3367	4825	64, 27
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
481	769	25, 12
4961	8161	81, 40
45	75	$1, \frac{1}{2} = 30$
1679	2929	48, 25
161	289	15, 8
1771	3229	50, 27
56	106	9, 5

Figure 5

we are on the right track, we note that in every such pair the numbers a and b are both 'nice', that is, factorable in terms of 2, 3, and 5. In five cases, the pattern breaks down and no pair exists. It will be a further confirmation if we can explain these discrepancies as errors made by the scribe who produced the tablet. We make a simple hypothesis and assume that B and C were each computed independently from the pair (a,b) and that a few errors were made but each affected only one number in each row. Thus in each vacant place we will assume that either B or C is correct and the other wrong, and attempt to restore the correct entry. Since we do not know the correct pair (a,b) we must find it; because of the evidence in the rest of the table, we insist that an acceptable pair must be composed of 'nice' sexagesimals.

We start with line 9; here, $B=541$, which happens to be the only prime in Column B. We therefore assume B is wrong and C is correct, and thus write $C=769=a^2+b^2$. This has a single solution, the pair (25, 12). (We also note that both happen to be nice sexagesimals.) If this is correct, then B should have been $(25)^2-(12)^2=481$, instead of 541 as given. Is there an obvious explanation for this mistake? Yes, for in slash notation, $541=9/1$ and $481=8/1$. The anomaly in line 9 seems to be merely a copy error.

Turn now to line 13; here, B is far larger than C, which is contrary to the pattern. Assume that B is in error and C is correct, and again try $C=289=a^2+b^2$. There is a 'nice' unique solution, (15, 8), and using these, we are led to conjecture that the correct value of B is $(15)^2-(8)^2=161$. Again, we ask if there is an obvious explanation for arriving at the incorrect value given, 25921. A partial answer is immediate: $(161)^2=25921$; so that for some reason the scribe recorded the *square* of the correct value for B.

Continuing, consider line 15. Since $B=56$ and $C=53$, we have $B>C$, which does not match the general pattern. However, it is not clear whether B is too large or C too small. Trying the first, we assume C is correct and solve $53=a^2+b^2$, obtaining the unique answer (7, 2). We reject this, since 7 is not a nice sexagesimal. Now assume that

B is correct, and write $56 = a^2 - b^2 = (a+b)(a-b)$. This has two solutions, (15, 13) and (9, 5). We reject the first and use the second, obtaining $9^2 + 5^2 = 106$ as the correct value of C. Seeking an explanation, we note that the value given by the scribe, 53, is exactly *half* of the correct value.

Turning now to line 2 of Figure 4; we have $B = 3367$ and $C = 11521$, either of which might be correct. Assume that $C = a^2 + b^2$ and find two solutions (100, 39) and (89, 60). While 100 and 60 are nice, 39 and 89 are not, so we reject both pairs and assume that B is correct. Writing $3367 = (a-b)(a+b)$ and factoring 3367 in all ways, we find four pairs: (1684, 1683), (244, 237), (136, 123), (64, 27), of which we can accept only the last. This yields $(64)^2 + (27)^2 = 4825$ as the correct C. Comparing this with the number 11521 that appeared on the tablet, we see no immediate naive explanation for the error. For example, since $4825 = 1/20/25$ and $11521 = 3/12/1$, it does not seem to be a copy error. Without an explanation, we may have a little less confidence in this reconstruction of the entries in line 2.

The last misfit in the table is line 11, where we have $B = 45$ and $C = 75$. This is unusual also because this is the only case where B and C have a common factor. The sums-and-differences-of-squares pattern failed because neither $C + B = 120$ nor $C - B = 30$ is twice a square. However, everything becomes clearer if we go back to base 60 notation and remember that we use floating point; for $120 = 2/0$, which is twice 1/0 and which we can also write as 1, clearly a perfect square. In the same way, 30 is twice 15, which is also 4^R and which is the square of 2^R. The pattern is preserved and no corrections need be made in the entries: with $a = 1 = 1/0$ and $b = \frac{1}{2} = 2^R = 30 = 0/30$, we have $a^2 = 1/0$ and $b^2 = 0/15$, and

$$C = a^2 + b^2 = 1/0 + 0/15 = 1/15 = 75$$

$$B = a^2 - b^2 = 1/0 - 0/15 = 0/45 = 45.$$

(Another aspect of the line 11 entries will appear later.)

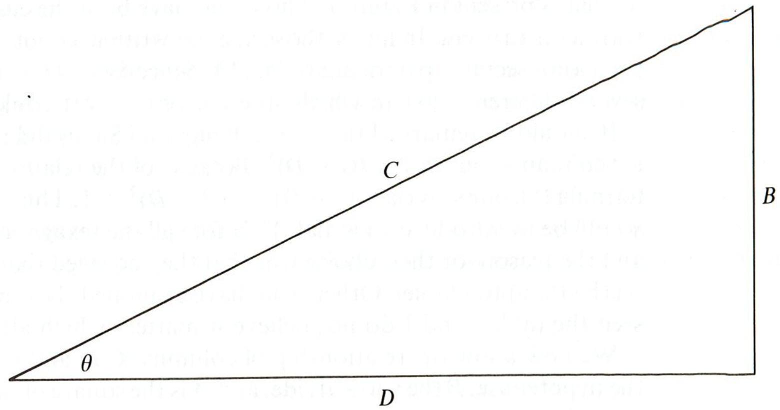
With this, we have completed the work of editing the original tablet. In Figure 5, we give a corrected table for columns B and C, together with the appropriate pairs (a, b) from which they can be calculated.

It is now the time to raise the second canonical question: What was the purpose behind this tablet? Speculation in this direction is less restricted, since the road is not as well marked. We can begin by asking if numbers of the form $a^2 - b^2$ and $a^2 + b^2$ have any special properties. In doing so, we run the risk of looking at ancient Babylonia from the twentieth century, rather than trying to adopt an autochthonous viewpoint. Nevertheless, one relation is extremely suggestive, involving both algebra and geometry. For any numbers (integers) a and b,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2. \qquad (*)$$

In addition, if we introduce $D = 2ab$, then B, C, and D can form a right-angled triangle with $B^2 + D^2 = C^2$. And finally, these formulas generate *all* Pythagorean triplets (triangles) from the integer parameters (a, b). (See Figure 6.)

There is no independent information showing that these facts were known to the Babylonians at the time we conjecture that this tablet was inscribed, although, as will appear later, their algebra had already mastered the solution of quadratic equations. If the tablet indeed is connected with this observation, then the unknown column A numbers ought to be connected in some way with the same triangle. The next step is,



$B = a^2 - b^2, \quad D = 2ab, \quad C = a^2 + b^2$

Figure 6

then, to proceed as before and try many different combinations of B, C, and D, in hopes that one of these will approximate the entries in column A. Slopes and ratios are an obvious starting point, so one calculates $C \div B$, $C \div D$, $B \div D$, etc. After discarding many failures, one arrives at the combination $(B \div D)^2$. In Figure 7, we give the values of this expression, calculated from the corrected values of B and using the hypothetical values of (a, b) to find D. (We remark that it was very helpful to have a programmable pocket calculator that could be trained to work in sexagesimal arithmetic!)

If we now return to Figure 1 and compare the numerals given there in column A with those that appear in Figure 7, we see that there is almost total agreement. For example, in line 10 we have exact duplication of an eight-digit sexagesimal! On probabilistic grounds alone, this is an overwhelming confirmation. Of course, at the top of the tablet where there were gaps due to erosion, Figures 1 and 7 are not the same, but it is evident that the calculated data in Figure 7 can be regarded as filling in the gaps. There are two minor disagreements in the two tables. In line 13, the tablet does not show an internal

Calculated Values of $(B \div D)^2$

line	
1	59/0/15
2	56/56/58/14/50/6/15
3	55/7/41/15/33/45
4	53/10/29/32/52/16
5	48/54/1/40
6	47/6/41/40
7	43/11/56/28/26/40
8	41/33/45/14/3/45
9	38/33/36/36
10	35/10/2/28/27/24/26/40
11	33/45
12	29/21/54/2/15
13	27/0/3/45
14	25/48/51/35/6/40
15	23/13/46/40

Figure 7

'0' that is present in Figure 7. This could have been the custom of the scribe in dealing with such an event. In line 8, the scribe has written a digit '59' where there should have been a consecutive pair of digits, '45/14'. Since $59 = 45 + 14$, it is not difficult to invent several different ways in which an error of this sort could have been made.

It should be remarked that Neugebauer and Sachs did not use $(B \div D)^2$ as a source for column A but rather $(C \div D)^2$. Because of the relationship between B and C, and formula (*), one sees that $(C \div D)^2 = (B \div D)^2 + 1$. Thus, the only effect of the change would be to introduce an initial '1/' before all the sexagesimals that appear in Figure 7, and the reason for their choice was that they believed that this was true for column A on the Plimpton tablet. Others who have examined the tablet do not agree. (I have not seen the tablet, and I do not believe it matters which alternative is used.)

We now know the relationship of columns A, B, and C. Referring to Figure 6, C is the hypotenuse, B the vertical side, and A is the square of the slope of the triangle; thus, in modern notation $A = \tan^2 \theta$. It is interesting to observe that the anomalous case of line 11, with $B = 45$ and $C = 75$, turns out to be the familiar 3, 4, 5 triangle; in the Babylonian case, this would seem to have been the $\frac{3}{4}, 1, \frac{5}{4}$ triangle, since $45 = 3 \times 4^R$ and $75 = 1/15 = 5 \times 4^R$. Of course the triangle, the side D, and the parameters (a, b) are all constructs of ours and not immediately visible in the original tablet. All that we can assert without controversy is that $A = B^2 \div (C^2 - B^2)$.

Let us reexamine some of our reasoning. In lines 2, 9, 13, and 15, the scribe recorded correct values for A but incorrect values for C, B, B, and C, respectively. This suggests strongly that A was not calculated directly from the values of B and C, but that A, B, and C were all calculated independently from data that do not appear on the tablet; our hypothetical pair (a, b) gains life. (Of course there is the possibility that the tablet before us is merely a copy from another master tablet.) In either case, it seems odd that column A should be error free while columns B and C, involving simpler numbers, should have four errors.

Other questions can be raised. If, as argued by Neugebauer (*The Exact Sciences in Antiquity*, Dover, 1969), the purpose of the tablet was to record a collection of integral-sided Pythagorean triangles (triplets), why do we not see the value of D, or at least the useful parameters (a, b)? And why would one want the values in column A which are squares of the slope? And why should the entries be arranged in an order that makes the numbers A decrease monotonically?

Variants of this explanation have been proposed. If one computes the values of the angle θ for each line of the tablet, they are seen to decrease steadily from about 45° to about 30°, in steps of about 1°. Is this an accident? Could this tablet be a primitive trigonometric table, intended for engineering or astronomic use? But again, why is $\tan^2 \theta$ useful?

Additional confirmation of such a hypothesis could be given by an outline of a computation procedure leading to the tablet, which makes all of the errors plausible and also shows why they would have occurred preferentially in columns B and C.

Building upon an earlier suggestion of Bruins, an intriguing explanation has been recently proposed by Voils. In Nippur, a large number of 'school texts' have been found, many containing arithmetic exercises. Among these, a standard puzzle problem is quite common. The student is given the difference (or sum) of an unknown number and its reciprocal and asked to find the number. If x is the number (called 'igi') and x^R is its reciprocal (called 'igibi'), then the student is to solve the equation $x - x^R = d$. Thus, the 'igi and igibi' problems are quadratic equations of a standard variety.

The school texts teach a specific solution algorithm: 'Find half of d, square it, add 1, take the square root, and then add and subtract half of d'. This is easily seen to be nothing more than a version of the quadratic formula, tailored to the 'igi and igibi' problems. Voils connects this class of problems, and the algorithm above, with the Plimpton tablet as follows.

First, assume with Bruins that the tablet was computed not from the pair (a, b) but from a single parameter, the number $x = a \div b$. Since a and b are both 'nice', the number x and its reciprocal x^R can each be calculated easily. Indeed, $x = a \times b^R$ and $x^R = b \times a^R$, and a^R and b^R, each appear in a standard reciprocal table. Next observe that

$$B = a^2 - b^2 = (ab)(x - x^R)$$

$$C = a^2 + b^2 = (ab)(x + x^R)$$

$$A = \left(\frac{B}{D}\right)^2 = \left\{\frac{1}{2}(x - x^R)\right\}^2.$$

This shows that the entries A, B, C in the Plimpton tablet could have been easily calculated from a special reciprocal table that listed the paired values x and x^R. Indeed, the numbers B and C can be obtained from $x \pm x^R$ merely by multiplying these by integers chosen to simplify the result and shorten the digit representation.

Voils adds to this suggestion of Bruins the observation that the numbers A are exactly the results obtained at the end of the second step in the solution algorithm, $(d/2)^2$, applied to an igi-igibi problem whose solution is x and x^R. Furthermore, the numbers B and C can be used to produce other problems of the same type but having the same intermediate results in the solution algorithm. Thus Voils proposes that the Plimpton tablet has nothing to do with Pythagorean triplets or trigonometry but, instead, is a pedagogical tool intended to help a mathematics teacher of the period make up a large number of igi-igibi quadratic equation exercises having known solutions and intermediate solution steps that are easily checked.

It is possible to point to another weak confirmation of this last approach. Suppose that we want a graduated table of numbers x and their reciprocals x^R. We start with the class of *all* pairs (a, b) of relatively prime integers such that $b < a < 100$ and each integer a and b is 'nice', factorable into powers of 2, 3, and 5. It is then easy to find the terminating Babylonian representation for both $x = a \div b$ and for $x^R = b \div a$. Make a table of these, arranged with x decreasing. Impose one further restriction:

$$\sqrt{3} < x < 1 + \sqrt{2}.$$

(This corresponds to the limitation $30° < \theta < 45°$, where θ is the base angle in the triangle in Figure 6.)

Then, the resulting list of pairs will coincide with that given in Figure 5, the corrected Plimpton table, except for three minor points. The pair (16, 9) does not appear, the pair (125, 54) does appear, and instead of the pair (2, 1) we have the pair $(1, \frac{1}{2})$; in passing, we recall that the last pair yields the standard 3, 4, 5 Pythagorean triangle.

Unlike Doyle's stories, this has no final resolution. Any of these reconstructions, if correct, throws light upon the degree of sophistication of the Babylonian mathematicians and breathes life into what was otherwise dull arithmetic. [...] I can do no better than to close with an analogy used by Neugebauer (*The Exact Sciences in Antiquity*, p. 177):

> In the 'Cloisters' of the Metropolitan Museum in New York there hangs a magnificent tapestry which tells the tale of the Unicorn. At the end we see the miraculous animal captured, gracefully resigned to his fate, standing in an enclosure surrounded by a neat little fence. This picture may serve as a simile for what we have attempted here. We have artfully erected from small bits of evidence the fence inside which we hope to have enclosed what may appear as a possible living creature. Reality, however, may be vastly different from the product of our imagination; perhaps it is vain to hope for anything more than a picture which is pleasing to the constructive mind, when we try to restore the past.

1.E3 Jöran Friberg on the purpose of Plimpton 322

We start by making the following simple but extremely important observation. With very few exceptions all Babylonian mathematical problem texts contain problems whose solutions are rational numbers or, more precisely, semiregular numbers which can be expressed by use of the Babylonian sexagesimal notation. It is evident that the authors of these Babylonian mathematical texts must have devoted a lot of work and ingenuity in *choosing* the right kind of *data* in their formulation of the problems, and in *devising problems* they knew would possess solutions of the indicated kind. For brevity, I call such problems *solvable*. For example, the problem of finding the third side of a right triangle when two of the sides are given becomes 'solvable' only if the sides of the given triangle are multiples of the sides of a primitive Pythagorean triangle with one of the shorter sides regular.

Thus it appears that the reason for the construction of the tables on the Plimpton tablet was not an interest in number-theoretical questions, but rather the need to *find data for a 'solvable' mathematical problem*. More precisely, it is my belief that the purpose of the author of Plimpton 322 was to write a *'teacher's aid' for setting up and solving problems involving right triangles*. In fact, a typical Babylonian problem text contains not only the formulation of the problem but also the details of its numerical solution for the given data. Hence the contents of the table on the (intact) Plimpton table would have given a teacher the opportunity to set up a large number of solved problems involving right triangles, with full numerical details, as well as to formulate a series of exercises for his students where only the necessary data were given, although the teacher *knew* that the problem was solvable, and where he could *check* the numerical details of the students' solutions by using the numbers in the table.