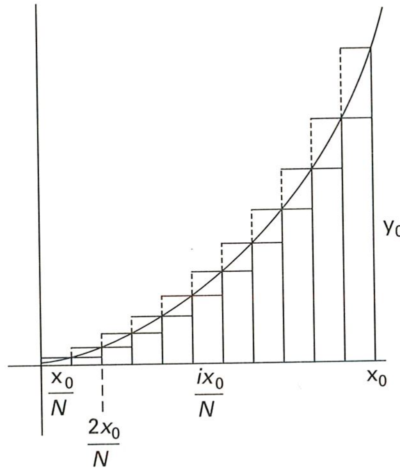

FIGURE 12.10

The area under $y = px^k$ according to Fermat and Roberval.

The obvious question then is how either of these two men discovered the formula for the sums of powers, a formula that was in essence known to ibn al-Haytham 600 years earlier. Fermat claimed that he had a "precise demonstration" and doubted that Roberval had one. In fact, as is typical in Fermat's work, all we have is his own general statement in terms of numbers, pyramidal numbers, and the other numbers that occur as columns in Pascal's triangle: "The last side multiplied by the next greater makes twice the triangle. The last side multiplied by the triangle of the next greater side makes three times the pyramid. The last side multiplied by the pyramid of the next greater side makes four times the triangulotriangle. And so on by the same progression in infinitum."[18] Fermat's statement, which we write as

$$N\binom{N+k}{k} = (k+1)\binom{N+k}{k+1},$$

is equivalent to Pascal's twelfth consequence. Using the properties of Pascal's triangle, it is then not difficult to derive for each k in turn (beginning with $k = 1$) an explicit formula for the sum of the kth powers. This formula will be of the form

$$\sum_{i=1}^{N} i^k = \frac{N^{k+1}}{k+1} + \frac{N^k}{2} + p(N)$$

where $p(N)$ is a polynomial in N of degree less than k. A careful study of the form of $p(N)$ enables one then to derive Roberval's inequality.

It is not known whether Fermat actually proved the general result indicated or merely tried a few values of k and assumed it would be true for any value. And it is also not known how Fermat derived formulas for the sums of powers of integers. Probably, Fermat was not aware of the work of Johann Faulhaber (1580–1635), a *Rechenmeister* from Ulm, who by 1631 had developed explicit formulas for the sums of kth powers of integers through $k = 17$.[19] And Pascal himself, writing in 1654, may not have been aware of Fermat's results either, when he showed an explicit derivation for sums of powers from properties of his

triangle and noted that "those who are even a little familiar with the doctrine of indivisibles will not fail to see that one may use this result for the determination of curvilinear areas. This result permits one immediately to square all types of parabolas and an infinity of other curves."[20]

In any case, Fermat was not completely satisfied with his method of finding areas because it only worked for higher parabolas. He could not see how to adapt it for curves of the form $y^m = px^k$ or for "higher hyperbolas" of the form $y^m x^k = p$. In modern terms, this method for finding areas under $y = px^k$ worked only if k were a positive integer. Fermat wanted a method that would work if k were any rational number, positive or negative. Although he only announced such a method in his *Treatise on Quadrature* of about 1658, it seems clear that he discovered this new procedure in the 1640s.

To apply his earlier method to the question of determining the area under $y = px^{-k}$ to the right of $x = x_0$ required dividing either the x-axis or the line segment $x = x_0$ from 0 to $y_0 = px_0^{-k}$ into finitely many intervals and summing the areas of the inscribed and circumscribed rectangles. Using the latter procedure, however, would give Fermat an infinite rectangle as the difference between his circumscribed and inscribed rectangles, one for which it was not at all clear that the area could be made as small as desired. On the other hand, there was no way of dividing the (infinite) x-axis into finitely many intervals ultimately to be made as small as one wishes. Fermat's solution to his dilemma was to divide the x-axis into infinitely many intervals, whose lengths were not equal but formed a geometric progression, and then to use the known formula for summing such a progression to add up the areas of the infinitely many rectangles.

Fermat began by partitioning the infinite interval to the right of x_0 at the points $a_0 = x_0$, $a_1 = \frac{m}{n}x_0$, $a_2 = \left(\frac{m}{n}\right)^2 x_0, \ldots, a_i = \left(\frac{m}{n}\right)^i x_0, \ldots$ where m and n ($m > n$) are positive integers (Fig. 12.11). The intervals $[a_{i-1}, a_i]$ will ultimately be made as small as desired by taking m/n sufficiently close to 1. Fermat next circumscribed rectangles above the curve over

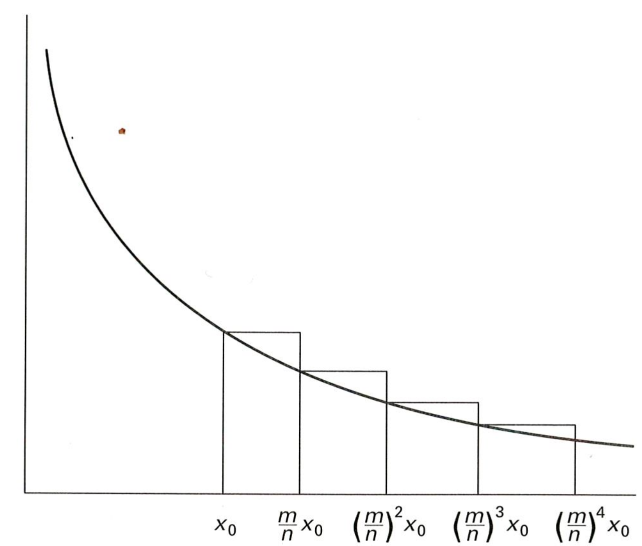

FIGURE 12.11

Fermat's procedure for determining the area under $y = px^{-k}$.