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**Proposition 9** $a(bc) = b(ac) = c(ab)$

**Proposition 10** Same with 4 numbers: $a(bcd) = bcd, a$ times, and $bcd = b(cd)$ so, using 9, $a(b(cd)) = b(a(cd)) = b(acd)$.

Generalise: "By the process of rising step by step without end, this is proved; that is, if one multiplies a number which is the product of four numbers by a fifth number, the result is the same as when one multiplies the product of any four of these by the other number. Therefore, the result of multiplying any product of numbers by another number contains any of these numbers as many times as the product of the others."

## Sum of Cubes

**Proposition 41** [Induction step] The square of the sum of the natural numbers from 1 up to a given number is equal to the cube of the given number added to the square of the sum of the natural numbers from 1 up to one less than the given number.

$$(1 + 2 + \cdots + n)^2 = n^3 + (1 + 2 + \cdots + (n-1))^2$$

Proof of
$(1+2+\cdots+n)^2 = n^3 + (1+2+\cdots+(n-1))^2$

$n^3 = n \cdot n^2$, and [P30:]
$n^2 = (1+2+\cdots n) + (1+2+\cdots+(n-1))$
So, $n^3 = n[(1+2+\cdots n) + (1+2+\cdots+(n-1))]$
$= n^2 + n[2(1+2+\cdots+(n-1))]$

$(1+2+\cdots+(n-1)+n)^2 =$
$n^2 + 2n(1+2+\cdots+(n-1))$
$+ (1+2+\cdots+(n-1))^2$

$n^3 + (1+2+\cdots+(n-1))^2 = (1+2+\cdots+n)^2$

## Gathering Induction

$$(1+2+\cdots+n)^2 = n^3 + (1+2+\cdots+(n-1))^2$$

1 has no precedent, but "it's third power is the square of the sum of the natural numbers up to it."

**Proposition 42** The square of the sum of the natural numbers from 1 up to a given number is equal to the sum of the cubes of the numbers from 1 up to the given number.

## Permutations

**Proposition 63** If the number of permutations of a given number of different elements is equal to a given number, then the number of permutations of a set of different elements containing one more number equals the product of the former number of permutations and the given next number.

If there are $P$ permutations of a given number of elements. Then there are $P$ with the new element first, and then the first element can be exchanged for any of the other elements to produce $P$ permutations for each of the original elements, for a total of $(n+1)P$ new permutations.

## Permutations

"Thus it is proved that the number of permutations of a given set of elements is equal to that number formed by multiplying together the natural numbers from 1 up to the number of given elements. For the number of permutations of 2 elements is 2, and that is equal to $1 \cdot 2$, the number of permutations of 3 elements is equal to the product $3 \cdot 2$, which is equal to $1 \cdot 2 \cdot 3$, and so one shows this result further without end."

**Proposition 65** If a certain number of elements is given and the number of permutations of order a number different from and less than the given number of elements is a third number, then the number of permutations of order one more in this given set of elements is equal to the number which is the product of the third number and the difference between the first and the second numbers.

"It has thus been proved that the permutations of a given order in a given number of elements are equal to that number formed by multiplying together the number of integers in their natural sequence equal to the given order and ending with the number of elements in the set."

## Final Results

**Proposition 66** $P_k^n = C_k^n P_k^k$

**Proposition 67** $C_k^n = \frac{P_k^n}{P_k^k}$

**Proposition 68** $C_k^n = C_{n-k}^n$

Consequence?