Infinite Limits & Limits at Infinity

Calculus I

Modeling Practices in Calculus

1 Infinite Limits

An **infinite limit** occurs when as the independent variable approaches a value, the dependent variable becomes arbitrarily large in magnitude. So, the function values increase or decrease without bound near a point.

Infinite limits are denoted by:

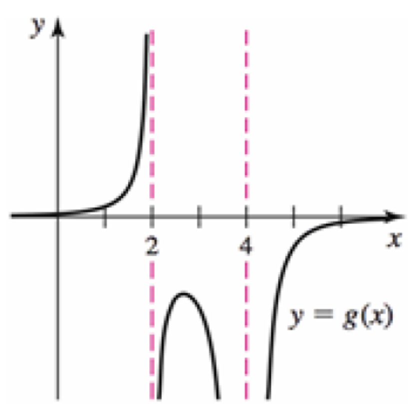
$$\lim_{x\to a} f(x) = \infty \text{ or } \lim_{x\to a} f(x) = -\infty$$

In the case of one-sided infinite limits:

$$\lim_{x\to a^-} f(x) = \infty \text{ or } \lim_{x\to a^-} f(x) = -\infty$$
$$\lim_{x\to a^+} f(x) = \infty \text{ or } \lim_{x\to a^+} f(x) = -\infty$$

Note: In the event of an infinite limit, the limit does not exist since as x approaches some number, the function values become increasingly large in magnitude, never approaching a single, unique value. However, we don't use the notation of "DNE" to express this since we have a formal way to describe this occurrence.

Example 1: Evaluate the following limits using the graph of $g(x)$ given below.



- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2} g(x)$
- $\lim_{x\to 4^+} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 4^-} g(x)$
- $\lim_{x\to 4} g(x)$

1.1 Finding Infinite Limits Analytically

Many infinite limits are analyzed using a common arithmetic property:

> The fraction $\frac{a}{b}$ grows arbitrarily large in magnitude if the denominator, b, approaches 0 while the numerator, a, remains nonzero and relatively constant. In other words, $\lim_{b\to 0} \frac{a}{b} = \pm\infty$.

Let's take the following limits for example:

$$\lim_{x\to 0^+} \frac{1}{x} \qquad \lim_{x\to 0^-} \frac{1}{x} \qquad \lim_{x\to 0} \frac{1}{x}$$

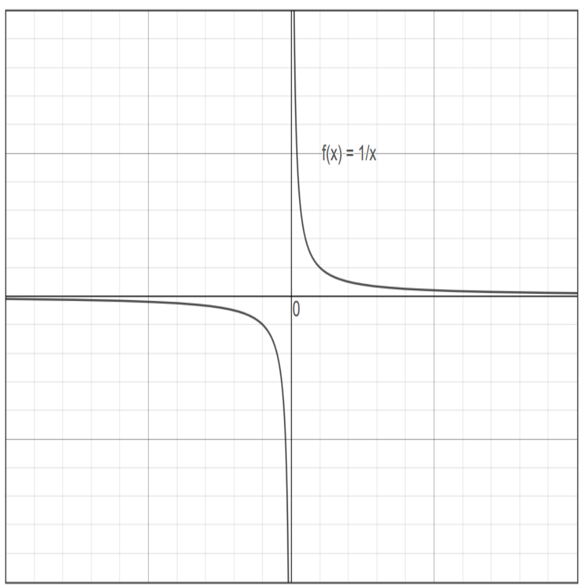
In all three cases, direct substitution does not work because we would be dividing a nonzero number by zero. So, let's observe what happens as we let x approach zero from either side. In the tables below, we can approach this numerically and by using the graph of $f(x) = \frac{1}{x}$, we can observe what happens graphically.

x approaches 0 from the right:

x	1	0.1	0.01	0.001	0.0001	0.00001
$\frac{1}{x}$	1	10	100	$1,000$	$10,000$	$100,000$

x approaches 0 from the left:

x	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
$\frac{1}{x}$	-1	-10	-100	$-1,000$	$-10,000$	$-100,000$



We see, as x approaches zero from the right and left, the function values grow larger in magnitude. Specifically, as x approaches zero from the right, the function values grow more and more positive, while as x approaches zero from the left, the function values grow more and more negative. So, we can draw the following conclusions for each limit:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0} \frac{1}{x} \text{ DNE}$$

Let's look at the left-hand limit $\left(\lim\limits_{x\to 0^-}\frac{1}{x}\right)$ and solve it **analytically** (by figuring out how the function behaves without actually plugging in several values or without having to look at its graph).

- Since x is approaching zero from the left, this means that the x-values will be less than zero. So, they will be negative.
- In the numerator, what x is approaching does not matter since 1 is a constant. [*Recall the first of our common limits in techniques for computing limits.*]
- For the denominator, x is getting closer to zero and is negative.
- So, overall, we have a positive constant being divided by an increasingly small negative number. The result of this fraction will be an increasingly large negative number.

This gives us the following:

$$\lim_{x\to 0^-}\frac{\overbrace{1}^{\text{approaches } 1}}{\underbrace{x}_{\text{approaches } 0,\text{ negative}}}=-\infty$$

Example 2: Evaluate each of the following limits:

- $\lim\limits_{x\to 1^+}\frac{\overbrace{-2}^{\to -2}}{\underbrace{x-1}_{\to 0,+}}=-\infty$

  Since we are approaching 1 from the right, this means that $x>1$. This implies $x-1>0$. So, $x-1$ will get closer to zero and will be positive as $x\to 1^+$. Therefore, we have a negative constant being divided by an increasingly small positive number. The result will be a number which gets larger in magnitude and negative. So, the limit will be $-\infty$.

- $\lim\limits_{x\to 1^-}\frac{-2}{x-1}$

- $\lim\limits_{x\to 2^-}\frac{x+1}{(x-2)^2}$

1.2 Vertical Asymptotes

For any given function $f(x)$ if any of the following limits are true, then $f(x)$ will have a **vertical asymptote** at $x = a$.

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a} f(x) = \pm\infty$$

Only one of the above limits has to occur in order for a function to have a vertical asymptote at $x = a$.

Example 3: Let's revisit the function $f(x) = \dfrac{1}{x}$.

In previous work, we found the following:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

From this we are able to draw the conclusion that the function $f(x) = \dfrac{1}{x}$ has a vertical asymptote at $x = 0$.

Finding Vertical Asymptotes of Rational Functions

Given a rational function, $\dfrac{p(x)}{q(x)}$, we can determine the vertical asymptotes, if any, with the following steps.

- Find the x-values where the denominator is zero, but the numerator is NOT zero.
  - Find $x = a$ such that $q(a) = 0$, but $p(a) \neq 0$
- Using the value(s) identified in the previous step, take one-sided limits of the function.
  - Find $\displaystyle\lim_{x \to a^+} \frac{p(x)}{q(x)}$ and/or $\displaystyle\lim_{x \to a^-} \frac{p(x)}{q(x)}$
- If either of the above limits go to $\pm\infty$, then your function has a vertical asymptote at $x = a$.

Example 4: Identify any vertical asymptotes of $f(x) = \dfrac{x^2 + x - 2}{x^2 - 4x + 3}$

2 Limits at Infinity

A **limit at infinity** occurs when the independent variable increases or decreases without bound. So, limits at infinity tell us how a function is behaving as its x-values get increasingly more positive or negative.

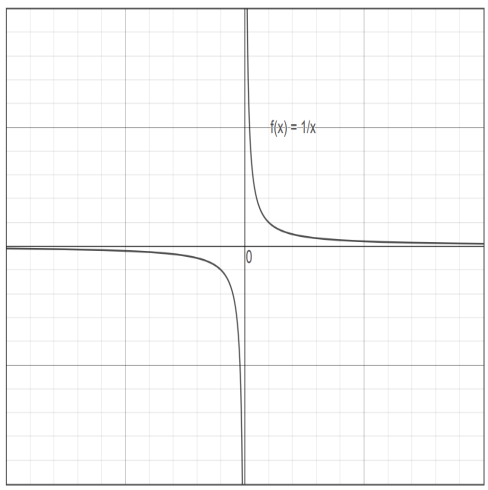
Limits at infinity are denoted by

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

Example 5: Evaluate $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$

We note that direct substitution is not a valid approach for evaluating these limits since we can't simply plug in the values of $\pm\infty$. So, there are a couple of other approaches we can take.

- **Graphically:** Let's look at the graph of $f(x) = \frac{1}{x}$ and observe what happens as the x-values grow larger in magnitude.



From the graph, we see that as x increases and decreases more and more, the function values get smaller and smaller in magnitude, approaching the value of zero. So, we are able to conclude:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x} = 0$$

- **Analytically:** Similar to analyzing infinite limits, we can analyze limits at infinity using an arithmetic property:

The fraction $\frac{a}{b}$ grows arbitrarily small in magnitude if the denominator, b, grows arbitrarily large in magnitude while the numerator, a, remains nonzero and relatively constant. In other words, $\lim_{b\to\pm\infty} \frac{a}{b} = 0$.

For $f(x) = \frac{1}{x}$, as $x \to \infty$ the numerator stays constant at 1 and the denominator grows larger in magnitude and positive. Therefore we have a positive constant being divided by an increasingly large positive number. The result of this fraction will be an increasingly small, positive number. Therefore, $\lim_{x\to\infty} \frac{1}{x} = 0$.

Now, as $x \to -\infty$, again the numerator stays constant at 1, but the denominator grows larger in magnitude and negative. So, we have a positive constant being divided by an increasingly large negative number. The result of this fraction will be an increasingly small, negative number. This means we have $\lim_{x\to-\infty} \frac{1}{x} = 0$.

2.1 Horizontal Asymptotes

For any given function $f(x)$ if either of the following limits exists, then $f(x)$ will have a **horizontal asymptote** at $y = L$.

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

Note: A horizontal asymptote can only occur when the limit at infinity yields a finite number.

Example 6: Let's again revisit the function of $f(x) = \frac{1}{x}$.

We just found that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$. From this we are able to conclude that $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$.

Example 7: Find the horizontal asymptotes of the following functions. *[Hint: Consider the limit laws.]*

- $f(x) = \frac{-2}{x}$
- $f(x) = \frac{3}{x} + 5$

2.2 Infinite Limits at Infinity

If a function $f(x)$ increases or decreases without bound as x increases or decreases without bound, then we have an infinite limit at infinity. These are denoted by:

$$\lim_{x\to\infty} f(x) = \pm\infty \text{ or } \lim_{x\to-\infty} f(x) = \pm\infty$$

Here are some examples of infinite limits at infinity:

- $\lim_{x\to\infty} x = \infty$
- $\lim_{x\to-\infty} x^2 = \infty$
- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to-\infty} x^3 = -\infty$

Note: In the case of infinite limits at infinity there exists no horizontal asymptotes.

2.3 Limits at Infinity of Powers and Polynomials

Let n be a positive integer and let p be the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

where $a_n \neq 0$.

- If n is even: $\lim\limits_{x\to\pm\infty} x^n = \infty$
- If n is odd: $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$
- $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = 0$
- $\lim\limits_{x\to\pm\infty} p(x) = \infty$ or $-\infty$, depending on the degree (odd or even?) of the polynomial and the sign of the leading coefficient a_n (positive or negative).
  - $\lim\limits_{x\to\pm\infty} p(x) = \lim\limits_{x\to\pm\infty} a_n x^n = a_n \cdot \lim\limits_{x\to\pm\infty} x^n$

Example 8: Evaluate the following:

- $\lim\limits_{x\to-\infty} (7x^5 - 4x^3 + 2x - 9) = \lim\limits_{x\to-\infty} 7x^5 = 7 \cdot \lim\limits_{x\to-\infty} x^5 = 7 \cdot -\infty = -\infty$
- $\lim\limits_{x\to\infty} (8x^2 + 3x - 5x^3)$
- $\lim\limits_{x\to-\infty} (17x^3 - 4x^9 - 5x + 1)$
- $\lim\limits_{x\to-\infty} \dfrac{-2}{x^3}$
- $\lim\limits_{x\to\infty} 18$

2.4 Limits at Infinity for Rational Functions

In general, for rational functions, we can evaluate limits at infinity by dividing each term in the expression by **the highest power of x in the denominator**, simplifying, and then evaluating the limit (for each term).

Example 9: Evaluate $\lim\limits_{x\to\infty} \dfrac{3x+2}{x^2-1}$

Since this is a limit at infinity, direct substitution is not a valid approach. However, given this is a rational function, we can divide each term by the highest power in the denominator. For this function, the highest power in the denominator is x^2. So, we have the following:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+2}{x^2-1} &= \lim_{x\to\infty} \frac{\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x}+\frac{2}{x^2}}{1-\frac{1}{x^2}} \\
&= \frac{\lim\limits_{x\to\infty} \dfrac{3}{x} + \lim\limits_{x\to\infty} \dfrac{2}{x^2}}{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \dfrac{1}{x^2}} \\
&= \frac{0+0}{1-0} \\
&= 0
\end{aligned}$$

For the example, since we found that a limit at infinity led to a finite value, we are able to say $f(x) = \dfrac{3x+2}{x^2-1}$ has a horizontal asymptote at $y = 0$.

Example 10: Evaluate $\lim\limits_{x\to\infty} \dfrac{x+x^3-8x^4}{2x^4+x^2-1}$

Example 11: Evaluate $\lim\limits_{x\to-\infty} \dfrac{x^5+4x^2-5}{x^3+6x}$