Special Limits & Continuity

Calculus I

Modeling Practices in Calculus

1 Special Limits

Here we present some limits that will be helpful in evaluating other limits. You'll want to keep these in mind as you work through certain types of limit problems.

- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to\infty} \ln(x) = \infty$
- $\lim_{x\to 0^+} \ln(x) = -\infty$
- $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$
- $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$
- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

Example 1: Evaluate the following: $\lim_{x\to 0^-} e^{\frac{1}{x}}$

Since we cannot divide by zero, direct substitution will not give a valid output. However, we can attempt to simplify this problem by rewriting the expression so that it resembles one of the special limits we have. The "complex" part of the expression is the exponent, $\frac{1}{x}$. Let's use a change of variable in order to simplify this piece and rewrite our limit in terms of the new variable.

$$\text{Let } t = \frac{1}{x}.$$

With this change of variable, we can rewrite the function we are taking the limit of to be in terms of t:

$$e^{\frac{1}{x}} = e^t$$

Now, we need to write the value the independent variable, x, is approaching in terms of our new variable, t. In our original limit, x is approaching 0 from the left. So, what does this mean for our new variable, t?

$$\lim_{x\to 0^-} t = \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Using an infinite limit, we see that as $x \to 0^-$, $t \to -\infty$. We can now rewrite our limit in terms of t and use one of our special limits to evaluate:

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to-\infty} e^t = 0$$

Example 2: Evaluate the following: $\lim_{x\to 2^+} \tan^{-1}\left(\frac{1}{x-2}\right)$

Example 3: Evaluate the following: $\lim_{x\to 0} \frac{\sin(5x)}{x}$

Example 4: Evaluate the following: $\lim_{x\to 0} \frac{2\sin(x)-\cos(x)+1}{x}$

[*Hint: This problem does not require a change of variable. Instead, try using algebra to rewrite the expression.*]

1.1 The Squeeze Theorem

The Squeeze Theorem

If the functions $f(x)$, $g(x)$, and $h(x)$ satisfy the following for all values of x near a:

$$f(x) \leq g(x) \leq h(x)$$

and if

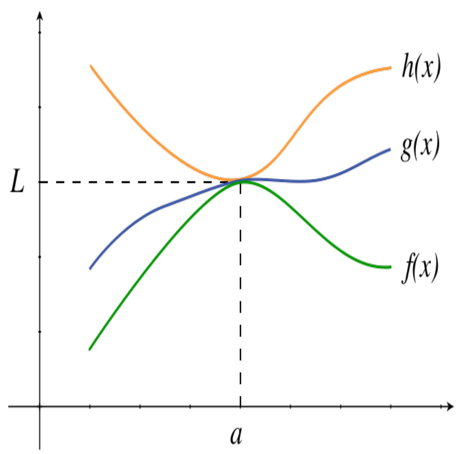
$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then

$$\lim_{x \to a} g(x) = L.$$

Note: The value a may be a finite number or $\pm\infty$.

A graphical example of the Squeeze Theorem is given below.



Example 5: Evaluate the following: $\lim_{x \to \infty} \frac{\sin(x)}{x}$

Using the methods for evaluating limits we've covered so far, we don't have a way to solve this limit. So, let's try to approach this using the Squeeze Theorem.

First, we note the following by using the known range of the sine function.

$$-1 \leq \sin(x) \leq 1$$

Since $x \to \infty$, it is reasonable to assume that $x > 0$. So, using properties of inequalities,

$$\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

Taking limits at infinity of the two outer functions we get:

$$\lim_{x \to \infty} \frac{-1}{x} = 0 \text{ and } \lim_{x \to \infty} \frac{1}{x} = 0$$

Therefore, by the Squeeze Theorem, we are able to conclude

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0.$$

Example 6: Given there exists a function, $f(x)$, such that $2x - 1 \leq f(x) \leq x^2$ for all x, find $\lim_{x \to 1} \frac{f(x) - 1}{x - 1}$.

2 Continuity

The graph of a function can be described as a "continuous curve" if it is "unbroken". In this section, we will explain the idea of an "unbroken curve" by using a mathematical property known as **continuity**. To make this idea more precise we need to develop some properties that make a curve "unbroken" or continuous.

2.1 Conditions for Continuity

A function $f(x)$ is **continuous** at a point $x = a$ if *each* of the following conditions are upheld:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If *one or more of these conditions are not upheld*, then $f(x)$ is **not continuous** at $x = a$ and $x = a$ is a **point of discontinuity**.

Example 7: Using the checklist for continuity, determine if $f(x)$ is continuous at $x = 3$.

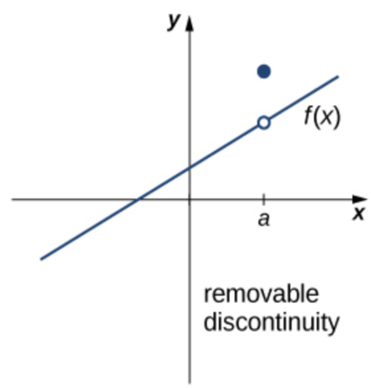
$$f(x) = \begin{cases} x^2 & : x < 3 \\ x + 6 & : x > 3 \\ 10 & : x = 3 \end{cases}$$

2.2 Types of Discontinuity

If a function $f(x)$ is not continuous at a number a, we say $f(x)$ has a **discontinuity** at $x = a$.

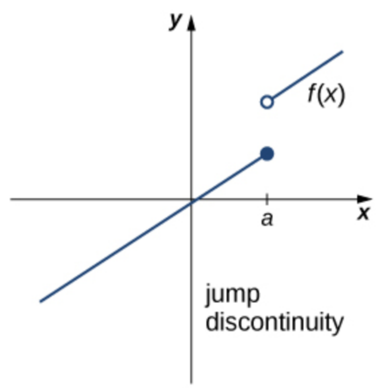
- **Removable discontinuity (hole)**:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x), \text{ but } \lim_{x\to a} f(x) \neq f(a)$$



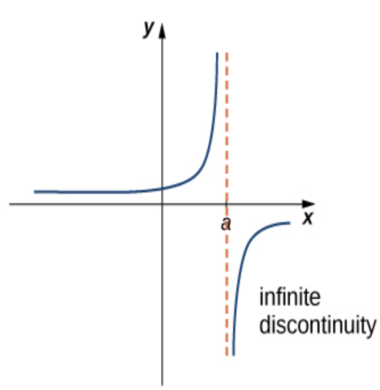
- **Jump Discontinuity**: The function "jumps" from one value to another at the point $x = a$.

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$



- **Infinite Discontinuity**: There is a vertical asymptote at $x = a$.

$$\lim_{x\to a} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$



To determine where a function is discontinuous, use similar rules for determining the domain of that function (we can't divide by zero, can't take the even root of a negative number, etc.). Then, if needed, use limits to determine the type(s) of discontinuity that occurs at these points.

Example 8: Determine if the following functions have points of discontinuity. If so, determine what type each are.

- $f(x) = \dfrac{3}{4-x}$

- $g(x) = \begin{cases} x^2 - 9 & : x \leq 3 \\ 2x - 5 & : x > 3 \end{cases}$

- $h(x) = \dfrac{x^2 - 25}{x - 5}$

2.3 Continuity Rules

If two functions, $f(x)$ and $g(x)$, are continuous at a point $x = a$, then the following are also continuous at $x = a$:

- $f(x) \pm g(x)$
- $cf(x)$
- $f(x) \cdot g(x)$
- $\frac{f(x)}{g(x)}$, as long as $g(a) \neq 0$
- $f(g(x))$, as long as f is continuous at $g(a)$
- $[f(x)]^n$, where n is a positive integer

These rules are important because we can use them to find that:

- Polynomial functions, $p(x)$, are continuous everywhere.
- Rational functions, $\frac{p(x)}{q(x)}$, are continuous everywhere EXCEPT where $q(x) = 0$.
- A function $[f(x)]^{n/m}$ with m *odd* is continuous at all points where $f(x)$ is continuous (m, n are positive integers with no common factors).
- A function $[f(x)]^{n/m}$ with m *even* is continuous at all points that f is continuous and $f(a) \geq 0$ (m, n are positive integers with no common factors).
- The trigonometric functions $\sin(x)$ and $\cos(x)$ are continuous everywhere, while the functions $\tan(x)$, $\sec(x)$, $\csc(x)$ and $\cot(x)$ are continuous at every point in their domain (remember that their domains are restricted).