Rates of Change as Limits & The Definition of the Derivative; Differentiability

Calculus I

Modeling Practices in Calculus

1 Rates of Change as Limits

We begin this section by revisiting rates of change. Recall that the **average rate of change** of a function f on the interval $[a, x]$ is the **slope of the secant line** between the two points x and a.

$$ARoC = m_{secant} = \frac{f(x) - f(a)}{x - a}$$

To find the **instantaneous rate of change** of f at a single point a, we find the **slope of the tangent line** at $x = a$. This can be accomplished by taking the limit of the slope of secant lines between the points x and a as x gets closer to a.

$$IRoC = m_{tangent} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

Another way of writing an interval between two points is $[a, a + h]$, where h represents the change in the x-values. So, the expressions for average and instantaneous rates of change becomes:

$$ARoC = m_{secant} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$$

$$IRoC = m_{tangent} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

We now have two ways of finding the instantaneous rate of change at a single point $x = a$ by using limits [Equations (1) and (2) above]. This also means we have two ways of calculating the slope of the tangent line at a point $x = a$. So, we are able to also find the **equation for the tangent line** to the graph of $f(x)$ at $x = a$. We can do so in point-slope form by:

$$y - f(a) = m_{tangent}(x - a)$$

Example 1: Let $f(x) = x^2 - 5x$.

- Find the slope of the tangent line to the graph of $f(x)$ at $x = 1$.

  Let's use Equation (2) for finding m_{tan}. Since we are finding the slope of the tangent at $x = 1$, this means in the equation $a = 1$. So, we have:

$$\begin{aligned} m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[(1+h)^2 - 5(1+h)] - (-4)}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 5 - 5h + 4}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 3h}{h} \\ &= \lim_{h \to 0} \frac{h(h-3)}{h} \\ &= \lim_{h \to 0} (h - 3) \\ &= -3 \end{aligned}$$

- Find the equation of the tangent line at $x = 1$.

$$\begin{aligned} y - f(a) &= m_{tan}(x - a) \\ y - f(1) &= m_{tan}(x - 1) \\ y - (-4) &= -3(x - 1) \\ y + 4 &= -3(x - 1) \leftarrow \textit{point-slope form} \\ y &= -3x - 1 \leftarrow \textit{slope-intercept form} \end{aligned}$$

Example 2: Let $f(x) = \dfrac{3}{x}$. Using Equation (1), find the slope and equation of the tangent line to the graph of $f(x)$ at $x = 2$.

Example 3: Find the equation of the tangent line to the graph of $f(x) = x^3 + 4$ at $x = -1$.

2 The Derivative Function

In the last section we showed that the limit: $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ can be interpreted as the slope of the tangent line to the curve $y = f(x)$ at the point $x = a$. Let's say we wanted to know the slope of the tangent line at any given point, x, on the curve. This is where what's known as the **derivative function** comes into play.

2.1 Limit Definition of the Derivative

The **derivative** of the function $f(x)$ with respect to x is the function $f'(x)$ defined by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of $f(x)$ represents the slope of the curve (slope of tangent line) at any given point, x, on $f(x)$.

2.2 Derivative Notation

For $y = f(x)$, the notation for the derivative function includes:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

For a specified point, $x = a$, the derivative of a function $y = f(x)$ at $x = a$ can be denoted as:

$$f'(a) = y'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}[f(x)]\right|_{x=a}$$

Example 4: Using the limit definition of the derivative, find the derivative of $f(x) = 3x^2 - 7x + 2$.

$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{[3(x+h)^2 - 7(x+h) + 2] - [3x^2 - 7x + 2]}{h} \\
&= \lim_{h\to 0} \frac{[3(x^2 + 2xh + h^2) - 7(x+h) + 2] - [3x^2 - 7x + 2]}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 7x - 7h + 2 - 3x^2 + 7x - 2}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 - 7h}{h} \\
&= \lim_{h\to 0} \frac{h(6x + 3h - 7)}{h} \\
&= \lim_{h\to 0} (6x + 3h - 7) \\
&= 6x + 3(0) - 7 \\
&= 6x - 7
\end{aligned}
$$

Example 5: Using the limit definition of the derivative, find the derivative of $f(x) = \dfrac{5}{x+1}$.

Example 6: Using the limit definition of the derivative, find the derivative of $f(x) = 8 - 2x$.

2.3 The Derivative at a Point

Using the limit definition of the derivative, it follows that the **derivative of a function evaluated at a point** $x = a$, $f'(a)$, is found by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note that this is the same equation we used to find the **slope of a tangent line at a point** $x = a$ [See: Equation (2)]. Therefore,

$$m_{tangent} = f'(a)$$

This also means that we can write the **equation of a tangent line** to the curve $y = f(x)$ at the point $x = a$ as:

$$y - f(a) = f'(a)(x - a)$$

Example 7: Find an equation of the tangent line to the curve $f(x) = 2\sqrt{x}$ at $x = 9$.

3 Differentiability

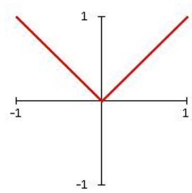
If the derivative, $f'(x)$, exists for any point or interval on $f(x)$, then we say that $f(x)$ is **differentiable** at that point or on that interval.

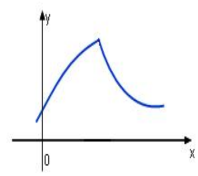
Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is guaranteed to be continuous at $x = a$.

Note: The converse of this theorem is not true. In other words, it is possible for a function to be continuous at a point, but not differentiable at that point.

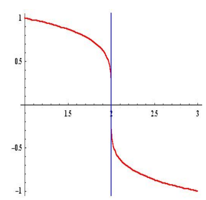
There are four cases where a function will be **non-differentiable**, or where the derivative will not exist.

1. **Corner/Sharp Turn:** A corner or sharp turn can be seen on the graph of a function. At this point where the corner or sharp turn exists, the function will not be differentiable because the one-sided derivatives will differ at this point.

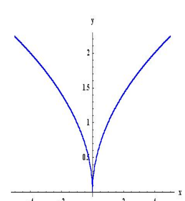




2. **Vertical Tangent:** At a vertical tangent the slope of the tangent line approaches ∞ or $-\infty$ from both sides.



3. **Cusp:** At a cusp the slope of the tangent line approaches ∞ from one side and $-\infty$ from the other.



4. **Discontinuity:** If a function is discontinuous at a point, then the derivative will not exist at that point.