Differentiation Rules; The Product and Quotient Rules; Derivatives of Trigonometric Functions

Calculus I

Modeling Practices in Calculus

1 Differentiation Rules

This section focuses on several rules that allow us to differentiate basic functions and their combinations without having to use the limit definition of the derivative each time.

Recall: There are several ways to denote the derivative of a function. For $y = f(x)$, the notation for the derivative function includes:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

Each of these indicate that we are taking the derivative of f or y with respect to x.

Basic Differentiation Rules

- **Constant Rule:** If c is a real number,

$$\frac{d}{dx}[c] = 0$$

- **Power Rule:** If n is a real number,

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

  - *Note:* $\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1x^{1-1} = 1x^0 = 1 \cdot 1 = 1$

- **Constant Multiple Rule:** If c is a real number,

$$\frac{d}{dx}[cf(x)] = c \cdot \frac{d}{dx}[f(x)] = cf'(x)$$

- **Sum/Difference Rule:** Given two differentiable functions, $f(x)$ and $g(x)$,

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] = f'(x) \pm g'(x)$$

Example 1: Find the derivative of the following:

- $f(x) = 5$

  Using the **constant rule**,

  $$f'(x) = \frac{d}{dx}[5] = 0$$

- $y = x^{10}$

  Using the **power rule**,

  $$\frac{dy}{dx} = \frac{d}{dx}[x^{10}] = 10x^{10-1} = 10x^9$$

- $g(x) = 15x$

  Using the **constant multiple rule** and **power rule**,

  $$\frac{dg}{dx} = \frac{d}{dx}[15x] = 15 \cdot \frac{d}{dx}[x] = 15 \cdot 1 = 15$$

- $h(x) = 6\sqrt[3]{x}$

- $y = x - 4x^2 + \sqrt{x} + \dfrac{1}{x} - 8$

Example 2: Let $f(x) = 2x^3 - 6x$.

- Find $f'(x)$.

- Find an equation of the line tangent to the graph of $f(x)$ at $x = -3$. [*Hint: Recall the relationship between the derivative and* m_{tan}.]

- For what value(s) of x does $f(x)$ have a slope of 12?

- For what value(s) of x does $f(x)$ have a horizontal tangent?

1.1 Higher-Order Derivatives

The importance and usefulness of higher-order derivatives will be more apparent later in this course and in later courses. But the idea of them is presented here as an introduction.

Higher-order derivatives involve computing derivatives multiple times. For example, if $f(x)$ has the derivative $f'(x)$, the derivative of $f'(x)$ is the **second derivative** of $y = f(x)$ and is denoted:

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2}[f(x)] = \frac{d}{dx}[f'(x)]$$

The number of times a function is differentiated is called the **order** of the derivative. In general, for positive integers n, the n^{th} **order derivative** is denoted:

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n}[f(x)] = \frac{d}{dx}[f^{(n-1)}(x)]$$

The value of a general n^{th} order derivative at a specific point $x = a$ is denoted:

$$f^{(n)}(a) = \left.\frac{d^n f}{dx^n}\right|_{x=a} = \left.\frac{d^n y}{dx^n}\right|_{x=a} = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=a}$$

Example 3: Find the second derivative of $f(t) = 3t + 2t^{11}$.

To find the second derivative, we must start by finding the first derivative, $f'(t)$, and then computing the derivative of $f'(t)$:

$$\begin{aligned} f'(t) &= 3 + 22t^{10} \\ f''(t) &= 220t^9 \end{aligned}$$

Example 4: Find the fifth derivative of $h(x) = 5x^4 - 3x^3 + 6x - 3$.

Example 5: Given $y = \frac{1}{x^2}$, find $\left.\frac{d^3 y}{dx^3}\right|_{x=2}$.

2 The Product and Quotient Rules

2.1 The Product Rule

Given two functions, $f(x)$ and $g(x)$, are differentiable at x, then the derivative of their product is found by:

$$\begin{aligned} \frac{d}{dx}\left[f(x)g(x)\right] &= \frac{d}{dx}[f(x)]g(x) + \frac{d}{dx}[g(x)]f(x) \\ &= f'(x)g(x) + g'(x)f(x) \end{aligned}$$

Example 6: Find the derivative of $p(x) = (3 - x^2)(x^3 - x + 1)$.

$$\begin{aligned} p'(x) &= \frac{d}{dx}[(3 - x^2)(x^3 - x + 1)] \\ &= \frac{d}{dx}[3 - x^2](x^3 - x + 1) + \frac{d}{dx}[x^3 - x + 1](3 - x^2) \\ &= -2x(x^3 - x + 1) + (3x^2 - 1)(3 - x^2) \end{aligned}$$

Example 7: Find the derivative of $y = \left(x^4 - \frac{4}{\sqrt{x}}\right)(\sqrt[4]{x} + 2x)$.

2.2 The Quotient Rule

Given two functions, $f(x)$ and $g(x)$, are differentiable at x, then the derivative of their quotient is found by:

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{\frac{d}{dx}[f(x)]g(x) - \frac{d}{dx}[g(x)]f(x)}{[g(x)]^2} \\ &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \end{aligned}$$

Example 8: Find the derivative of $y = \frac{5x + 2}{3x^2 - x}$.

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\left[\frac{5x + 2}{3x^2 - x}\right] &= \frac{\frac{d}{dx}[5x + 2](3x^2 - x) - \frac{d}{dx}[3x^2 - x](5x + 2)}{(3x^2 - x)^2} \\ &= \frac{5(3x^2 - x) - (6x - 1)(5x + 2)}{(3x^2 - x)^2} \end{aligned}$$

Example 9: Find the derivative of $h(x) = \dfrac{7x+1}{2\sqrt{x}-3}$.

3 Derivatives of Trigonometric Functions

The objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions. We can accomplish this by using previous derivative rules and identities to find these derivatives. Let's start by using the limit definition of the derivative to find the derivative of $\sin(x)$ and $\cos(x)$.

Hint: You'll find the sine/cosine addition formulas and the special trigonometric limits helpful.

Find the first derivative of the following:

- $f(x) = \sin(x)$

$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\cdot \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}
$$

- $f(x) = \cos(x)$

The derivatives of the remaining trigonometric functions are given below.

Derivatives of Trigonometric Functions

- $\frac{d}{dx}[\sin x] = \cos x$
- $\frac{d}{dx}[\sec x] = \sec x \tan x$
- $\frac{d}{dx}[\tan x] = \sec^2 x$
- $\frac{d}{dx}[\cos x] =$
- $\frac{d}{dx}[\csc x] = -\csc x \cot x$
- $\frac{d}{dx}[\cot x] = -\csc^2 x$

Example 10: Find the first derivative of the following:

- $f(x) = x^2 \tan x$

- $y = \frac{1-x}{\csc x}$

Example 11: Find the second derivative of $f(x) = \sec x$.