The Chain Rule; Derivatives of Exponential, Logarithmic & Inverse Trigonometric Functions

Calculus I

Modeling Practices in Calculus

1 The Chain Rule

So far, we have ways for finding derivatives of combinations of functions including the sum, difference, product, and quotient. In this section, we will discuss how to evaluate derivatives of composite functions.

1.1 The Chain Rule - Leibniz Notation

Given a composite function $y = f(g(x))$, if we let $u = g(x)$, then $y = f(u)$. The derivative of y is then found by:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[y] \cdot \frac{d}{dx}[u] \\ &= \frac{d}{du}[f(u)] \cdot \frac{d}{dx}[g(x)] \end{aligned}$$

Using **Leibniz Notation for the chain rule** can be thought of in the following steps:

Given a composite function $y = f(g(x))$, in order to find $\frac{dy}{dx}$,

1. Identify the outer function, f, and the inner function, g.
2. Let $u = g(x)$ and $y = f(u)$.
3. Find the product $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
4. Put u back in terms of x.

Example 1: Find the derivative of $y = \cos(x^3)$.

We start by noticing that y is a composite function made up of the two functions, $f(x) = \cos(x)$ and $g(x) = x^3$. So,

$$y = f(g(x)) = \cos(x^3)$$

We'll let $u = g(x)$ and express y as $f(u)$:

$$\begin{aligned} u &= g(x) = x^3 \\ y &= f(u) = \cos(u) \end{aligned}$$

Let's proceed by finding the product for $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\cos(u)] \cdot \frac{d}{dx}[x^3] \\ &= -\sin(u) \cdot 3x^2 \\ &= -\sin(x^3) \cdot 3x^2 \\ &= -3x^2 \sin(x^3) \end{aligned}$$

Example 2: Using Leibniz Notation, find the derivative of the following:

- $y = (4 + 5x)^6$

- $y = \sqrt{\tan x}$

1.2 The Chain Rule - Prime Notation

Given a composite function $y = (f \circ g)(x) = f(g(x))$, the derivative of this function is:

$$y' = f'(g(x)) \cdot g'(x)$$

Using **Prime Notation for the chain rule** can be thought of in the following steps:

Given a composite function $y = (f \circ g)(x) = f(g(x))$, in order to find y',

1. Identify the outer function, $f(x)$, and the inner function, $g(x)$.

2. Take the derivative of the outer function, $f'(x)$, and evaluate it at the inner function: $f'(g(x))$.

3. Multiply by the derivative of the inner function: $f'(g(x)) \cdot g'(x)$

Example 3: Find the derivative of $y = \sin^2(x)$.

It may help to rewrite y so that it is easier to see how the function is composed.

$$y = \sin^2(x) = (\sin x)^2$$

Here, we notice that $y = f(g(x)) = (\sin x)^2$ is a composite function made up of the two functions $f(x) = x^2$ and $g(x) = \sin x$.

Now, let's take take the derivative of the outer function, $f(x) = x^2$:

$$f'(x) = 2x$$

and evaluate it at the inner function, $g(x) = \sin x$:

$$f'(g(x)) = f'(\sin x) = 2\sin x$$

Finally, let's multiply the result by the derivative of the inner function:

$$\begin{aligned} g'(x) &= \cos x \\ f'(g(x)) \cdot g'(x) &= 2\sin x \cdot \cos x \end{aligned}$$

So, our derivative of $y = \sin^2(x)$ is:

$$y' = f'(g(x)) \cdot g'(x) = 2\sin x \cos x$$

Example 4: Using Prime Notation, find the derivative of $y = \sec(x^2 + 3x)$.

Example 5: Find the derivative of the following:

- $h(x) = \dfrac{1}{(2x^3 + x^2 + 8)^5}$

- $y = x^4 \cos(4x^2)$

1.3 Composition of 3 or More Functions

When a function is composed of three or more functions, we must do the chain rule repeatedly to find the derivative. For example, if we are given that

$$y = f(g(h(x)))$$

Then,

$$\frac{dy}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Example 6: Find the derivative of $y = \sqrt[4]{\sin(3x - 2)}$.

2 Derivatives of Exponential, Logarithmic, & Inverse Trigonometric Functions

2.1 Derivatives of Exponential Functions

- $\frac{d}{dx}[b^x] = \ln(b)b^x$
- $\frac{d}{dx}\left[e^x\right] = e^x$

Example 7: Find the derivative of $f(x) = 3^x - 5e^x$.

Example 8: Find the derivative of $y = e^{\cot x}$.

2.2 Derivatives of Logarithmic Functions

- $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

Example 9: Find the derivative of $g(x) = 3x^2\log_3(x)$.

Example 10: Find the derivative of $y = \ln(x^2 + 5)$.

2.3 Derivatives of Inverse Trigonometric Functions

- $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}\left[\cos^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$, for $-\infty < x < \infty$
- $\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}$, for $-\infty < x < \infty$
- $\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}\left[\csc^{-1} x\right] = \frac{-1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

Example 11: Find the derivative of $y = \frac{1+x^2}{\cot^{-1} x}$.

Example 12: Find the derivative of $h(x) = \sec^{-1}(2x)$.