Implicit Differentiation

Calculus I

Modeling Practices in Calculus

1 Implicit Differentiation

Usually the equations we work with are presented in the form $y = f(x)$. An equation of this form is in **explicit form** because it is *explicitly* solved for in terms of one variable. For example, the function

$$y = x^2 + 2x + 1$$

is in explicit form because y is explicitly defined as a function of one variable, x.

However, when we have equations like the following:

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2y - 2 &= 3y^3 \end{aligned}$$

these exhibit an *implicit* relation between the variables. For equations of this form, **implicit form**, we don't have a straight-forward way to compute a y-value given an x-value. However, for equations in **implicit form**, a value of x can determine *one or more* values of y.

For equations in implicit form, **implicit differentiation** is used to find the derivative.

Steps for Implicit Differentiation

Given an equation in implicit form, in terms of x and y, in order to find $\frac{dy}{dx}$:

1. Take the derivative of both sides of the equation with respect to x.
   - $\frac{d}{dx}[\qquad\quad] = \frac{d}{dx}[\qquad\quad]$
2. Using the appropriate derivative rules, differentiate both sides of the equation. For the terms involving y, apply the chain rule.
   - $\frac{d}{dx}[y] = \frac{dy}{dx}$
3. Using appropriate algebra, solve for $\frac{dy}{dx}$.
   - This step usually involves putting all terms with $\frac{dy}{dx}$ on one side of the equation and the other terms on the other side.
   - If needed, you can then factor out $\frac{dy}{dx}$ in order to solve for it.

Example 1: Find $\frac{dy}{dx}$ for the implicitly defined curve $x^2 + y^2 = 1$.

We are trying to find $\frac{dy}{dx}$, or the derivative of y with respect to x. So, we start by taking the derivative of both sides of the equation with respect to x:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[1]$$

Now, we take the derivative (with respect to x) of each term, using appropriate rules and the chain rule for terms that involve y.

$$\begin{aligned} \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] &= \frac{d}{dx}[1] \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

Using algebra, we finish by solving for the derivative, $\frac{dy}{dx}$:

$$\begin{aligned} 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Example 2: Find $\frac{dy}{dx}$ for $x^2y - 2 = 3y^3$.

$$\begin{aligned} \frac{d}{dx}[x^2y - 2] &= \frac{d}{dx}[3y^3] \\ \text{*}\textit{product rule} \rightarrow 2x \cdot y + \frac{dy}{dx} \cdot x^2 - 0 &= 9y^2\frac{dy}{dx} \\ 2xy + x^2\frac{dy}{dx} &= 9y^2\frac{dy}{dx} \\ x^2\frac{dy}{dx} - 9y^2\frac{dy}{dx} &= -2xy \\ \frac{dy}{dx}\left(x^2 - 9y^2\right) &= -2xy \\ \frac{dy}{dx} &= \frac{-2xy}{x^2 - 9y^2} \end{aligned}$$

Example 3: Find $\frac{dy}{dx}$ for the following:

- $x^2 + \cos(y) = 3x - 4y$

- $e^{xy} = x - y$

Example 4: For $x^3 + x^2y + 4y^2 = 6$, determine the equation of the tangent line at the point $(1, 1)$.

1.1 Higher-Order Implicit Differentiation

Similar to the way we found higher-order derivatives for explicitly defined functions, we can do the same for implicitly defined curves. In this course, we are mainly only interested in the second derivative for implicitly defined curves, so we'll focus on that here.

Recall: If $\frac{dy}{dx}$ represents the first derivative of y with respect to x, then we find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$$

When finding higher-order derivatives for implicitly-defined curves, we use similar steps to finding the first derivative by using implicit differentiation.

Example 5: Find $\frac{d^2y}{dx^2}$ for $y^2 - 2x = 1 - 2y$.

Let's start by finding the first derivative, $\frac{dy}{dx}$:

$$\begin{aligned}
\frac{d}{dx}[y^2 - 2x] &= \frac{d}{dx}[1 - 2y] \\
2y\frac{dy}{dx} - 2 &= -2\frac{dy}{dx} \\
2y\frac{dy}{dx} + 2\frac{dy}{dx} &= 2 \\
\frac{dy}{dx}(2y + 2) &= 2 \\
\frac{dy}{dx} &= \frac{2}{2y+2} \\
\frac{dy}{dx} &= \frac{1}{y+1}
\end{aligned}$$

Now, we can find the second derivative by taking the derivative with respect to x of $\frac{dy}{dx}$:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[\frac{1}{y+1}\right] \\
\frac{d^2y}{dx^2} &= \frac{0\cdot(y+1) - \frac{dy}{dx}\cdot 1}{(y+1)^2} \quad \leftarrow \textit{quotient rule} \\
&= \frac{-\frac{dy}{dx}}{(y+1)^2}
\end{aligned}$$

Note:* We want to express derivatives (of any order) in terms of the variables used to define the original function or curve. So, for this example, we want the second derivative to be in terms of x and/or y **only. There is a straight-forward solution to this – simply replacing $\frac{dy}{dx}$ with what we found it to be in the previous step.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}}{(y+1)^2} \\ &= \frac{-\frac{1}{y+1}}{(y+1)^2} \\ &= \frac{-1}{(y+1)^3}\end{aligned}$$

Example 6: Find $\dfrac{d^2y}{dx^2}$ for $2x^3 - 3y^2 = 8$.

1.2 Logarithmic Differentiation

For "complex" functions involving a combination of products, quotients, or powers, derivatives can be taken with **logarithmic differentiation** by using properties of the natural logarithm and implicit differentiation.

Steps for Logarithmic Differentiation

Given a function $y = f(x)$, in order to find $\dfrac{dy}{dx}$:

1. Take the ln of both sides of the equation.
2. Simplify the result by using properties of the natural logarithm.
   - $\ln(xy) = \ln(x) + \ln(y)$
   - $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$
   - $\ln(x^p) = p\ln(x)$
3. Take the derivative of both sides (with respect to x) using implicit differentiation.
4. Solve for the derivative, $\dfrac{dy}{dx}$.
5. Substitute $y = f(x)$ back into the result.

Example 7: Find $\dfrac{dy}{dx}$ for $y = \dfrac{x\sqrt[3]{2x-7}}{(x^3+1)^4}$.

To find the derivative of this function, it is possible to use a combination of the quotient, product, and chain rules. However, that will make things pretty messy and complicated. So, we use logarithmic differentiation in order to simplify the process.

1. Take the ln of both sides of the equation.

$$\ln(y) = \ln\left(\frac{x\sqrt[3]{2x-7}}{(x^3+1)^4}\right)$$

2. Simplify the result by using properties of the natural logarithm.

$$\begin{aligned}
\ln(y) &= \ln\left(\frac{x\sqrt[3]{2x-7}}{(x^3+1)^4}\right) \\
&= \ln\left(x\sqrt[3]{2x-7}\right) - \ln\left((x^3+1)^4\right) \\
&= \ln(x) + \ln\left(\sqrt[3]{2x-7}\right) - \ln\left((x^3+1)^4\right) \\
&= \ln(x) + \frac{1}{3}\ln(2x-7) - 4\ln(x^3+1)
\end{aligned}$$

3. Take the derivative of both sides using implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}\left[\ln(x) + \frac{1}{3}\ln(2x-7) - 4\ln(x^3+1)\right] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{3}\cdot\frac{1}{2x-7}\cdot 2 - 4\cdot\frac{1}{x^3+1}\cdot 3x^2 \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{2}{3(2x-7)} - \frac{12x^2}{x^3+1}
\end{aligned}$$

4. Solve for the derivative, $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{2}{3(2x-7)} - \frac{12x^2}{x^3+1}\right)$$

5. Substitute $y = f(x)$ back into the result.

$$\frac{dy}{dx} = \frac{x\sqrt[3]{2x-7}}{(x^3+1)^4}\left(\frac{1}{x} + \frac{2}{3(2x-7)} - \frac{12x^2}{x^3+1}\right)$$

Example 8: Find $\dfrac{dy}{dx}$ for $y = \sqrt{(x^2+1)(x-1)^2}$

Note: Any function of the form $y = f(x)^{g(x)}$ *(a function of* x *raised to another function of* x*) requires logarithmic differentiation to find* $\frac{dy}{dx}$.

Example 9: Find the derivative of the following functions.

- $y = x^x$

$$\begin{aligned} \ln(y) &= \ln\left(x^x\right) \leftarrow \textit{Step 1} \\ &= x\ln(x) \leftarrow \textit{Step 2} \\ \frac{d}{dx}\left[\ln(y)\right] &= \frac{d}{dx}[x\ln(x)] \leftarrow \textit{Step 3} \\ \frac{1}{y}\frac{dy}{dx} &= 1\cdot\ln(x) + \frac{1}{x}\cdot x \\ \frac{dy}{dx} &= y\left(\ln(x)+1\right) \leftarrow \textit{Step 4} \\ &= x^x(\ln(x)+1) \leftarrow \textit{Step 5} \end{aligned}$$

- $y = x^{\cos x}$

- $y = (\sin x)^{x^2}$