Related Rates Problems

Calculus I

Modeling Practices in Calculus

In many physical problems, there are several quantities that all change with time. Often these rates of change are related to each other by the physical constraints of the system. In other words, as one quantity changes, other quantities change also. Related rates problems involve finding rates of change given other rates of change. These rates of change are usually related by time.

Solving Related Rates Problems

1. Read the problem carefully and make a sketch to **diagram** the given information.

2. **Label and define** the quantities in the problem – assign variable names to all relevant quantities and **explicitly** define each variable.

3. Write the **given quantities and rates** in terms of these variables.

4. Determine what rate the problem is asking you to find – **rate in question**.

5. Write an **equation(s)** that relates the variables in the problem. This equation is usually known by prior knowledge.

6. Use **implicit differentiation (with respect to time)** to find a relationship between the rates. (Take the derivative of both sides of the equation(s) with respect to time.)

7. **Substitute** in the given/known values of rates and quantities.

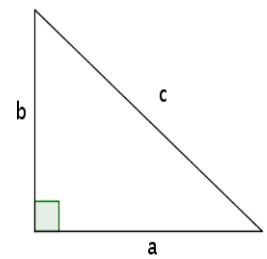
8. **Solve** for the rate you are interested in (what the problem is asking for); include units.

9. Double-check that your answer is **reasonable** in context of the problem.

10. Write a **summary sentence** – answer the original question in context of the problem.

Example 1: Two boats leave a port at the same time. Boat A travels east at 24 miles per hour, while Boat B travels north at 10 mph. At what rate is the distance between the two boats changing at the moment Boat A has traveled 12 miles and Boat B has traveled 5 miles?

1. Read the problem carefully and make a sketch to **diagram** the given information.



2. **Label and define** the quantities in the problem – assign variable names to all relevant quantities and **explicitly** define each variable.

$$\begin{aligned} a &= \text{distance between Boat A and the port} \\ b &= \text{distance between Boat B and the port} \\ c &= \text{distance between Boat A and Boat B} \\ t &= \text{time} \end{aligned}$$

3. Write the **given quantities and rates** in terms of these variables.

   We are interested in the moment in time when Boat A is 12 miles from the port and Boat B is 5 miles from the port. We are told that Boat A is traveling at 24 mph and Boat B at 10 mph. These values represent the rate at which the distance between each boat and the port (quantities a and b) is changing with respect to time. So,

$$\begin{aligned} a &= 12 \text{ miles} \\ b &= 5 \text{ miles} \\ \frac{da}{dt} &= 24 \text{ mph} \\ \frac{db}{dt} &= 10 \text{ mph} \end{aligned}$$

4. Determine what rate the problem is asking you to find – **rate in question**.

   We are asked to find the rate at which the distance between the two boats is changing. In context of the quantities and variables we have defined, we are trying to find how c is changing with respect to time. So, we want to find $\frac{dc}{dt}$.

5. Write an **equation(s)** that relates the variables in the problem. This equation is usually known by prior knowledge.

   By referencing the diagram above, we see that the relationship between the distances in this problem is represented by a right triangle. So, the equation that will relate the variables we have defined is a familiar one – the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

6. Use **implicit differentiation (with respect to time)** to find a relationship between the rates. (Take the derivative of both sides of the equation(s) with respect to time.)

$$\begin{aligned} \frac{d}{dt}[a^2 + b^2] &= \frac{d}{dt}[c^2] \\ 2a\frac{da}{dt} + 2b\frac{db}{dt} &= 2c\frac{dc}{dt} \end{aligned}$$

7. **Substitute** in the given/known values of rates and quantities.

   With the values we were given $\left(a, b, \frac{da}{dt}, \frac{db}{dt}\right)$, we are able to substitute these into the above equation. However, there's one quantity we don't have explicitly stated, c. Since we know the values for a and b for the moment in time we are interested in, we can find the value for c at this time by using the equation we identified:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 12^2 + 5^2 &= c^2 \\ 144 + 25 &= c^2 \\ 169 &= c^2 \\ 13 &= c \end{aligned}$$

   Now, we can substitute in all of our given/known values:

$$\begin{aligned} 2a\frac{da}{dt} + 2b\frac{db}{dt} &= 2c\frac{dc}{dt} \\ 2(12)(24) + 2(5)(10) &= 2(13)\frac{dc}{dt} \end{aligned}$$

8. **Solve** for the rate you are interested in (what the problem is asking for); include units.

$$\begin{aligned} 576 + 100 &= 26\frac{dc}{dt} \\ 676 &= 26\frac{dc}{dt} \\ \frac{dc}{dt} &= 26 \text{ mph} \end{aligned}$$

   Note: The units on $\frac{dc}{dt}$ *will be mph since it represents the rate at which the distance between the boats are changing.*

9. Double-check that your answer is **reasonable** in context of the problem.

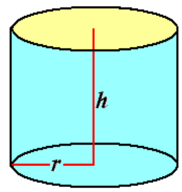
   This answer is reasonable for this problem since the rate we found is positive. This means the distance between the two boats should be getting larger, or increasing, at this time. This makes sense given the boats should be moving further apart from each other.

10. Write a **summary sentence** – answer the original question in context of the problem.

   At the moment Boat A has traveled 12 miles and Boat B has traveled 5 miles, the distance between the two boats is changing at rate of 26 mph.

Example 2: Suppose a cylindrical tank containing water has a leak which causes the water to drain at a rate of 50 in^3/s. If the tank has a radius of 10 inches, how is the height of the water changing when the tank is 25 inches full?

Let's make a sketch of what the system will look like and label the relevant quantities:



Define variables:

$$\begin{aligned} r &= \text{radius of circular base of cylinder} \\ h &= \text{height of water in tank} \\ V &= \text{volume of water in tank} \\ t &= \text{time} \end{aligned}$$

Given rates and quantities:

$$\begin{aligned} r &= 10 \text{ in} \\ h &= 25 \text{ in} \\ \frac{dV}{dt} &= -50 \text{ in}^3/\text{s} \textit{ (Why negative?)} \\ \frac{dr}{dt} &= 0 \text{ in/s} \textit{ (Why 0?)} \end{aligned}$$

Want to find:

$$\frac{dh}{dt}$$

Equation: volume of a cylinder

$$V = \pi r^2 h$$

Implicit differentiation with respect to time, substitute known values, and solve for wanted rate:

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}[\pi r^2 h] \\ \frac{dV}{dt} &= 2\pi r \frac{dr}{dt} \cdot h + \frac{dh}{dt} \cdot \pi r^2 \leftarrow \textit{product rule} \\ -50 &= 2\pi(10)(0)(25) + \frac{dh}{dt} \cdot \pi(10)^2 \\ -50 &= 100\pi \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{-50}{100\pi} = -\frac{1}{2\pi} \text{ in/s} \end{aligned}$$

It makes sense for our answer to be negative. Since water is leaking out of the tank, the height of the water should be decreasing. So, when the tank is 25 inches full, the height of the water is decreasing at a rate of $\frac{1}{2\pi}$ in/s.

Example 3: An offshore rig begins to spill oil in a circular patch centered on the rig. If the radius of the oil spill increases at a rate of 30 m/hr, how fast is the area of the oil spill increasing when the radius is 100 meters?

Example 4: One end of a 13 ft ladder is on the ground and the other end rests on a vertical wall. The bottom end of the ladder is drawn away from the wall at 3 ft/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 feet from the wall?

Example 5: Suppose that a balloon is being filled with air at a rate of 10 cm^3/s. (Assume that the balloon is a perfect sphere.) At what rate is the surface area of the balloon changing when the radius is 5 cm?