Absolute and Relative Extrema

Calculus I

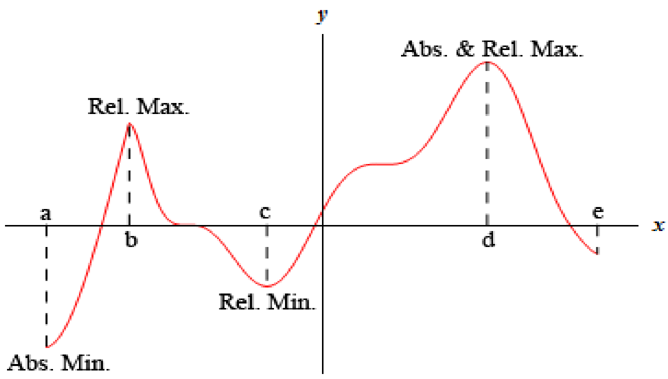
Modeling Practices in Calculus

An important application of the derivative is using it to find optimal (or best) solutions to problems. Optimization problems are found in various fields including mathematics, physical sciences, economics, and engineering. In this section we discuss how to apply derivatives to find extreme values of functions.

1 Absolute and Relative Extrema

- A function $f(x)$ has an **absolute (or global) maximum** value of $f(a)$ if $f(a) \geq f(x)$ for all x in the domain of f.
  - The largest function value over the domain of f.
- A function $f(x)$ has an **absolute (or global) minimum** value of $f(a)$ if $f(a) \leq f(x)$ for all x in the domain of f.
  - The smallest function value over the domain of f.
- A function $f(x)$ has a **relative (or local) maximum** value of $f(c)$ if $f(c) > f(x)$ for all x *near* $x = c$.
- A function $f(x)$ has a **relative (or local) minimum** value of $f(c)$ if $f(c) < f(x)$ for all x *near* $x = c$.
- The maxima and minima of f are called the **extreme values** of f.
  - Extreme values are the ***function values***, or the y-values.
  - The x-values give the ***location*** of the extrema, or where the extreme values occur.
- Relative extrema do **NOT** occur at endpoints of intervals.

Example 1: Given the graph of $f(x)$ below, identify the location of its absolute and relative extrema.

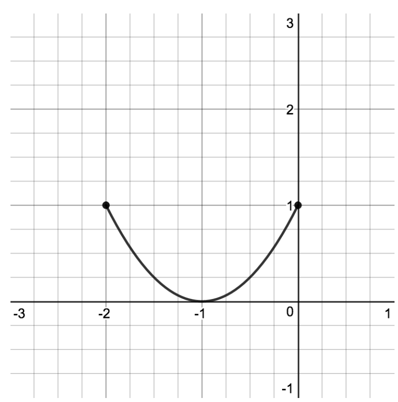


- The largest function value of $f(x)$ occurs at $x = d \Rightarrow$ ***Absolute maximum at*** $x = d$.
- The smallest function value of $f(x)$ occurs at $x = a \Rightarrow$ ***Absolute minimum at*** $x = a$.
- Both $x = b$ and $x = d$ are the locations of largest function values on the graph in the interval around each point $\Rightarrow$ ***Relative maxima at both*** $x = b$ ***and*** $x = d$.
- $x = c$ is the location of the smallest function value on the graph in the interval around that point $\Rightarrow$ ***Relative minimum at*** $x = c$.
- $x = a$ and $x = e$ are endpoints, so they cannot be a relative minimums.

Example 2: Identify the absolute and relative extrema for the following function on the given interval:

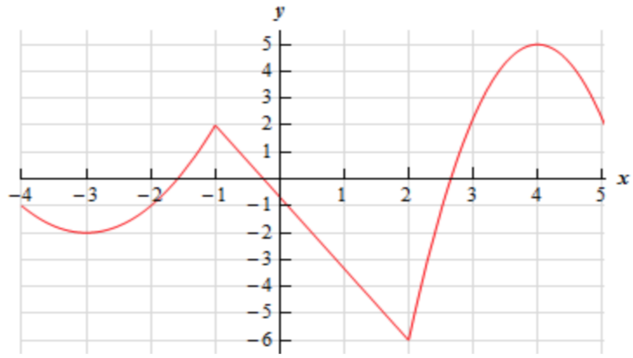
$$f(x) = (x + 1)^2;\ [-2, 0]$$

Here is the graph for this function on the given interval.



- There is an absolute minimum of 0 at $x = -1$.
- There is a relative minimum of 0 at $x = -1$.
- There is an absolute maximum of 1 at $x = -2$ and $x = 0$.
- There are no relative maximums.

Example 3: Below is the graph of some function $f(x)$. Identify all relative and absolute extreme values of $f(x)$ and their locations.



- Absolute maximum of _____ at _____
- Absolute minimum of _____ at _____
- Relative maximum of _____ at _____
- Relative maximum of _____ at _____
- Relative minimum of _____ at _____
- Relative minimum of _____ at _____

1.1 Finding Absolute Extrema on a Closed Interval

Extreme Value Theorem: If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is guaranteed to have an absolute maximum and absolute minimum somewhere on the interval $[a, b]$.
Note: A function does not have to be continuous to have extreme values.

Critical Points

A **critical point** of a function f is an ***interior point*** c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined.

- Critical points cannot occur at endpoints of a domain or interval.
- Relative extrema can only occur at critical points.
- Absolute extrema can only occur at critical points or at endpoints of an interval.

Finding Absolute Extrema of a Function f on a Closed Interval $[a, b]$

1. Find all critical points on the interval (a, b).
   - Find where $f'(x) = 0$.
   - Find where $f'(x)$ is undefined.
2. Find the *function values* at these critical points.
3. Find the *function values* at the endpoints, a and b.
4. The largest function value (found in Steps 2 and 3) is the absolute maximum value; the smallest function value (found in Steps 2 and 3) is the absolute minimum value.

Example 4: Find the absolute extrema of $f(x) = x^2 - 1$ on $[-1, 2]$.

1. Find all critical points on the interval (a, b).

   To find the critical points, we need the first derivative of $f(x)$.

   $$f'(x) = 2x$$

   Now, we find where $f'(x) = 0$ and where $f'(x)$ is undefined.

$f'(x) = 0$	$f'(x)$ undefined
$2x = 0$	$2x$ is a polynomial and is defined everywhere
$x = 0$	

   Since $x = 0$ is on the interval $(-1, 2)$:

   Critical points: $x = 0$

2. Find the *function values* at these critical points.

   Let's plug the critical point we just found into our given function, $f(x)$:

   $$f(0) = 0^2 - 1 = 0 - 1 = -1$$

3. Find the *function values* at the endpoints, a and b.

   Now, let's plug the endpoints of the given interval into $f(x)$:

   $$f(-1) = (-1)^2 - 1 = 1 - 1 = 0$$
   $$f(2) = 2^2 - 1 = 4 - 1 = 3$$

4. The largest function value is the absolute maximum value; the smallest function value is the absolute minimum value.

   Of the three function values we found in Steps 2 and 3, -1 is the smallest and 3 is the largest. So,

   There is an absolute maximum of 3 at $x = 2$.

   There is an absolute minimum of -1 at $x = 0$.

Example 5: Find the absolute extrema of $f(x) = (x^2 - 1)^{2/3}$ on $[-3, 1]$.

Find $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{2}{3}(x^2-1)^{-1/3} \cdot 2x \\ &= \frac{4x}{3(x^2-1)^{1/3}} \end{aligned}$$

Find critical points:

$\underline{f'(x) = 0}$

$$\frac{4x}{3(x^2-1)^{1/3}} = 0$$
$$4x = 0$$
$$x = 0$$

$\underline{f'(x) \text{ undefined}}$

$$\frac{4x}{3(x^2-1)^{1/3}} \text{ undefined}$$
$$3(x^2-1)^{1/3} = 0$$
$$x^2 - 1 = 0$$
$$x = 1,\ x = -1$$

Critical points: $x = 0$, $x = -1$

Note that $x = 1$ is an endpoint, so it cannot be a critical point.

Now, let's check the function values at both critical points and both endpoints.

$$\begin{aligned} f(0) &= (0-1)^{2/3} = (-1)^{2/3} = 1 \\ f(-1) &= (1-1)^{2/3} = 0^{2/3} = 0 \\ f(-3) &= (9-1)^{2/3} = 8^{2/3} = 4 \\ f(1) &= (1-1)^{2/3} = 0^{2/3} = 0 \end{aligned}$$

So, the smallest function value is 0 and the largest is 4. Therefore, our absolute extrema are:

Absolute maximum of 4 at $x = -3$.
Absolute minimum of 0 at $x = -1$ and $x = 1$.

Example 6: Find the absolute extrema of the following functions on the given intervals.

- $f(x) = \sqrt{4 - x^2}$; $[-1, 2]$
- $g(x) = \sin(x) - \cos(x)$; $[0, \pi]$

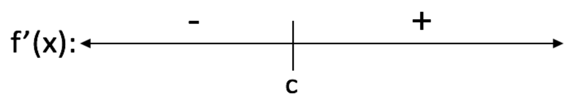
1.2 Finding Relative Extrema

Graphically, it can be pretty straightforward to be able to identify the locations of relative extrema. However, we should have a way to accomplish this without having to look at a graph. Recalling that relative extrema can only occur at critical points of a function, we can use a test to determine whether or not a critical point is a location of a relative extrema.

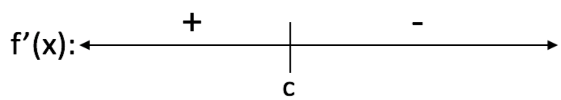
First Derivative Test

Suppose that $f(x)$ has a critical point at $x = c$.

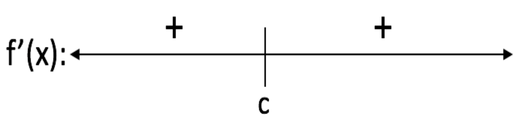
- If $f'(x)$ goes from ***negative-to-positive*** as we pass through c, from left to right, then $f(c)$ is a **relative (or local) minimum**.

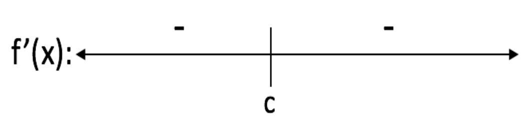


- If $f'(x)$ goes from ***positive-to-negative*** as we pass through c, from left to right, then $f(c)$ is a **relative (or local) maximum**.

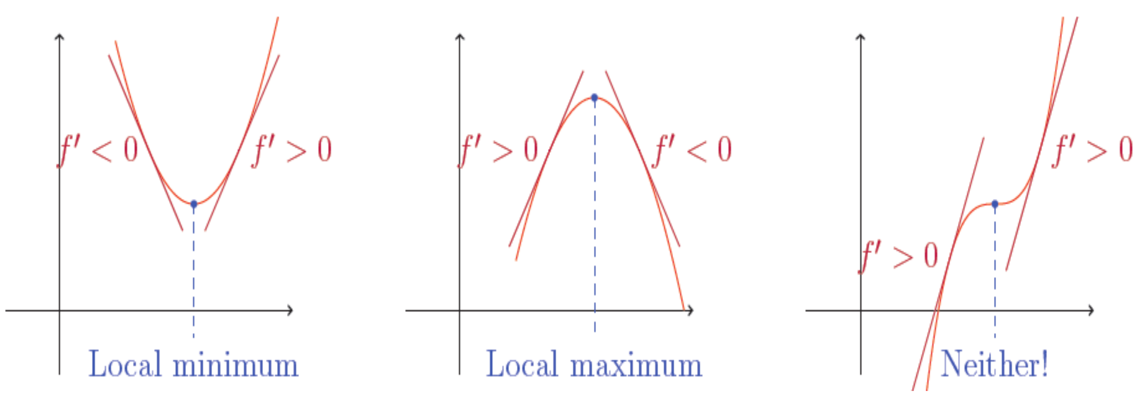


- If $f'(x)$ does not change sign as we pass through $x = c$ (negative or positive on both sides), then $f(c)$ is neither a relative minimum nor maximum.





We can see what each of these looks like graphically below:



Example 7: Find the relative extrema of the function $f(x) = \frac{1}{5}x^5 - 3x^3$.

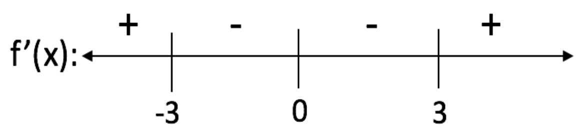
We start by finding the critical points of $f(x)$.

$$f'(x) = x^4 - 9x^2$$

$f'(x) = 0$	$f'(x)$ undefined
$x^4 - 9x^2 = 0$	$f'(x)$ exists everywhere
$x^2(x^2 - 9) = 0$	
$x^2 = 0;\ x^2 - 9 = 0$	
$x = 0,\ x = -3,\ x = 3$	

Critical points: $x = 0$, $x = -3$, $x = 3$

To determine if any of these critical points are location of relative extrema we conduct the first derivative test, illustrated by the below sign chart.



We see that $f'(x)$ goes from **positive-to-negative** as we pass through $x = -3$ which means this is the location of a **relative maximum**. And $f'(x)$ goes from *negative-to-positive* as we pass through $x = 3$, so it is the location of a *relative minimum*. Also, we see the derivative does not change sign as we pass through $x = 0$, so it is neither a relative minimum nor maximum.

So, the relative extrema for $f(x)$ are:

$$\text{Relative maximum of } f(-3) = \frac{162}{5} \text{ at } x = -3$$
$$\text{Relative minimum of } f(3) = -\frac{162}{5} \text{ at } x = 3$$

Example 8: Find the relative extrema of the function $f(x) = e^{(x^3 - 3x^2)}$.

1.3 Finding Absolute Extrema on Open Intervals

Suppose $f(x)$ is continuous and has **EXACTLY ONE** relative extrema on an interval at $x = x_0$.

- If $f(x)$ has a *relative minimum* at x_0, then $f(x_0)$ is the **absolute minimum** on the interval.
- If $f(x)$ has a *relative maximum* at x_0, then $f(x_0)$ is the **absolute maximum** on the interval.

Example 9: Find the absolute extrema, if any, of the function $f(x) = x^2e^x$ on the interval $(-1, \infty)$.

Since we are finding extrema, we start by finding the critical points.

$$f'(x) = 2xe^x + x^2e^x$$

$\underline{f'(x) = 0}$

$$2xe^x + x^2e^x = 0$$
$$xe^x(2 + x) = 0$$
$$xe^x = 0;\ 2 + x = 0$$
$$x = 0,\ x = -2$$

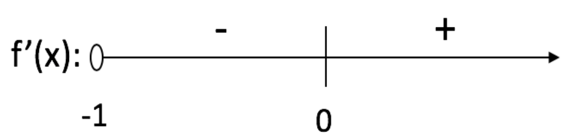
$\underline{f'(x) \text{ undefined}}$

$f'(x)$ exists everywhere

Critical points: $x = 0$

We note that $x = -2$ is not on the given interval, so it cannot be a critical point.

Since we have an open interval, we can't compare the function values of the critical point and endpoints. So, we use the first derivative test to determine if $x = 0$ is a relative extrema.



The first derivative test shows us there is a relative minimum at $x = 0$. Since this is the **only relative extrema of $f(x)$ on the given interval**, $x = 0$ must also be the location of the absolute minimum. So:

$$\text{Absolute minimum of } f(0) = 0^2 \cdot e^0 = 0 \text{ at } x = 0$$

Example 10: Find the absolute extrema, if any, of the function $f(x) = \dfrac{\ln(x)}{x}$ on the interval $(0, \infty)$.