How Derivatives Affect the Shape of Graphs; Summary of Curve Sketching

Calculus I

Modeling Practices in Calculus

Graphing utilities can be very useful for determining the shape of a graph. However, sometimes it can be difficult to determine key features of a graph by just using a graphing utility. The purpose of this section is to present mathematical processes that can be used to determine the exact shape and key features of a graph.

1 How Derivatives Affect the Shape of Graphs

1.1 Increasing/Decreasing

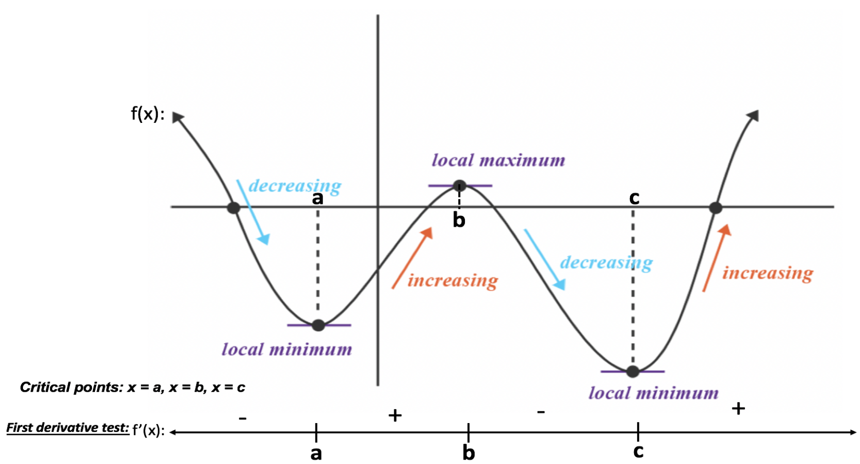
When sketching the graph of a function, it's important to know where the function increases and where it decreases. We can use the first derivative to determine this.

Increasing/Decreasing

Given a function $f(x)$:

- For all x where $f'(x) > 0$, $f(x)$ is increasing.
- For all x where $f'(x) < 0$, $f(x)$ is decreasing.
- For all x where $f'(x) = 0$, $f(x)$ is neither increasing or decreasing $\Rightarrow f(x)$ is constant

In order to determine where the first derivative is positive or negative, we can use critical points and the first derivative test.



Example 1: Find the critical points of $f(x) = x^3 - 3x^2 + 1$ and determine whether they are locations of relative maxima, relative minima, or neither. Also determine the intervals where $f(x)$ is increasing and decreasing.

We start by finding the critical points of $f(x)$.

$$f'(x) = 3x^2 - 6x$$

$\underline{f'(x) = 0}$

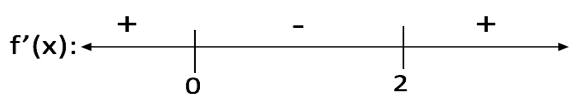
$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$3x = 0;\ x - 2 = 0$$
$$x = 0,\ x = 2$$

$\underline{f'(x) \text{ undefined}}$

$f'(x)$ exists everywhere

Critical points: $x = 0$, $x = 2$

To find the locations of relative extrema, we use the first derivative test.



From the first derivative test, we see the locations of relative extrema for $f(x)$ are:

Relative maximum at $x = 0$
Relative minimum at $x = 2$

To determine the intervals where $f(x)$ is increasing and decreasing, we can use the sign chart above. From the chart we see that $f'(x)$ is positive when $x < 0$ and $x > 2$, and $f'(x)$ is negative when $0 < x < 2$. So,

$f(x)$ ***is increasing on*** $(-\infty, 0)$ ***and*** $(2, \infty)$
$f(x)$ ***is decreasing on*** $(0, 2)$

Example 2: For the given function, find any critical points and determine the location(s) of any relative extrema. Also determine the intervals on which the function is increasing or decreasing.

$$f(x) = x^2e^x$$

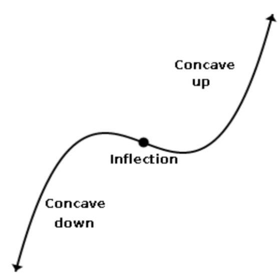
1.2 Concavity

Another key feature of a graph of a function is its *curvature*. For functions that are not constant or linear, their graphs have a curvature known as ***concavity***. On intervals where a function has **upward curvature**, we say the function is **concave up**. On intervals where the function has *downward curvature*, we say the function is *concave down*. ***We use the second derivative of a function to tell us about the concavity of the function.***

Concavity and Inflection Points

Given a function $f(x)$:

- For all x where $f''(x) > 0$, $f(x)$ is concave up → *upward u-shape*
- For all x where $f''(x) < 0$, $f(x)$ is concave down → *downward u-shape*
- The point where concavity changes is known as an ***inflection point***.
  - Possible inflection points are given by a point $x = a$ where $f''(a) = 0$ or $f''(a)$ is undefined.
  - If $f''(x)$ changes from *negative-to-positive* or *positive-to-negative* at $x = a$, then $(a, f(a))$ is a **point of inflection**.



Example 3: Find the intervals on which the function $f(x) = x^3 - \frac{1}{2}x^4$ is concave up and concave down. Also identify any point(s) of inflection.

When we are determining the concavity of a function, we use the second derivative. So, we start by finding $f''(x)$.

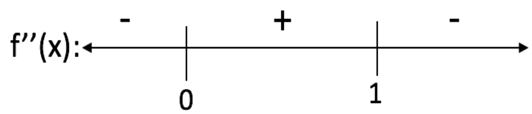
$$f(x) = x^3 - \frac{1}{2}x^4$$
$$f'(x) = 3x^2 - 2x^3$$
$$f''(x) = 6x - 6x^2$$

Similar to how we found critical points using the first derivative, **we find possible inflection points by using the second derivative**. So, we have to check where $f''(x) = 0$ and where $f''(x)$ is undefined.

$\underline{f''(x) = 0}$	$\underline{f''(x) \text{ undefined}}$
$6x - 6x^2$	$f''(x)$ exists everywhere
$6x(1 - x) = 0$	
$6x = 0;\ 1 - x = 0$	
$x = 0,\ x = 1$	

Possible inflection points: $x = 0$, $x = 1$

Now, we can use a sign chart with the possible inflection points we just found to determine where $f''(x)$ is positive or negative.



From the sign chart we see that $f''(x)$ is negative when $x < 0$ and $x > 1$ and positive when $0 < x < 1$. So,

$$f(x) \textbf{\textit{ is concave down on }} (-\infty, 0) \textbf{\textit{ and }} (1, \infty)$$
$$f(x) \textbf{\textit{ is concave up on }} (0, 1)$$

The sign chart for $f''(x)$ also shows us that as we pass through $x = 0$, $f''(x)$ goes from negative-to-positive. And as we pass through $x = 1$, $f''(x)$ goes from positive-to-negative. So, there are sign changes for $f''(x)$ at both points which means the function changes concavity at each of these points.

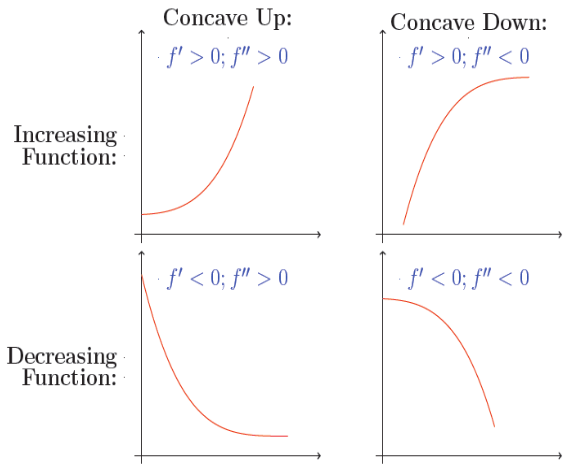
$$f(x) \textbf{\textit{ has a point of inflection at }} x = 0 \rightarrow \textbf{\textit{ Point: }} (0, 0)$$
$$f(x) \textbf{\textit{ has a point of inflection at }} x = 1 \rightarrow \textbf{\textit{ Point: }} \left(1, \frac{1}{2}\right)$$

Example 4: For the given function identify the intervals on which it is concave up and concave down. Also identify any points of inflection.

$$f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3$$

2 Summary of Curve Sketching

Using our knowledge of the first and second derivative, we see that there are four "generic" shapes for the behavior of functions between critical points and inflection points.



General Process for Graphing a Function

1. **Domain:** Determine the domain of $f(x)$. This will tell you what $x-$values, if any, are locations of discontinuities.
2. **Intercepts:** Find the $x-$ and $y-$intercepts of $f(x)$.
    - $x-$intercept: Set $f(x) = 0$ and solve for $x \to x$-int:$(x, 0)$
    - $y-$intercept: Find $f(0) \to y$-int:$(0, y)$
3. **Critical points:** Find all $x-$values such that $f'(x) = 0$ or $f'(x)$ is undefined.
4. **Increasing/Decreasing; Relative Extrema:** Make a sign chart for $f'(x)$ and find the intervals where the function is increasing/decreasing and the locations of relative extrema. (*You'll need $y-$values for the relative extrema for graphing these points.*)
5. **Possible Inflection Points:** Find all $x-$values such that $f''(x) = 0$ or $f''(x)$ is undefined.
6. **Concave Up/Concave Down; Inflection Points:** Make a sign chart for $f''(x)$ and find the intervals where the function is concave up/down and the location(s) of inflection points. (*You'll need $y-$values for the points of inflection for graphing these points.*)
7. **Asymptotes/End Behavior:** Determine all (if any) asymptotes, vertical and horizontal.
    - **Horizontal Asymptotes/End Behavior**: Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$
    - **Vertical Asymptotes**: If $f(x)$ is rational, find where a vertical asymptote may occur.
8. **Sketch $f(x)$:** Make your sketch using all of the preceding information.
    - Draw asymptotes as dashed lines.
    - Plot (and label) intercepts, relative maxima/minima points, and inflection points.
    - Draw the curve in between these points using the increasing/decreasing and concavity information.

Example 5: Sketch the graph of $f(x) = x^3 - 3x$.

1. **Domain:** $f(x)$ is a polynomial, so it exists everywhere $\Rightarrow$ ***Domain:*** $(-\infty, \infty)$
2. **Intercepts:**

x–intercept(s)	y–intercept
$x^3 - 3x = 0$ $x(x^2 - 3) = 0$ $x = 0,\ x = \sqrt{3},\ x = -\sqrt{3}$ ***x–int:*** $(0,0)$, $(\sqrt{3},0)$, $(-\sqrt{3},0)$	$f(0) = 0^3 - 3(0) = 0$ ***y–int:*** $(0,0)$

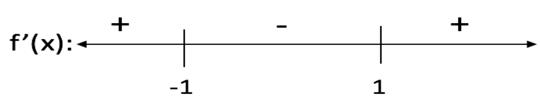
3. **Critical Points:**

$$f'(x) = 3x^2 - 3$$

$f'(x) = 0$	$f'(x)$ undefined
$3x^2 - 3 = 0$ $3(x^2 - 1) = 0$ $x = 1,\ x = -1$	$f'(x)$ is a polynomial and is defined everywhere

Critical points: $x = -1$, $x = 1$

4. **Increasing/Decreasing; Relative Extrema:**



$f(x)$ ***is increasing on*** $(-\infty, -1)$ ***and*** $(1, \infty)$
$f(x)$ ***is decreasing on*** $(-1, 1)$
There is a relative maximum at $x = -1 \Rightarrow$ ***Rel. max at the point*** $(-1, 2)$
There is a relative minimum at $x = 1 \Rightarrow$ ***Rel. min at the the point*** $(1, -2)$

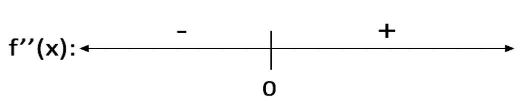
5. **Possible Inflection Points:**

$$f''(x) = 6x$$

$f''(x) = 0$	$f''(x)$ undefined
$6x = 0$ $x = 0$	$f''(x)$ is a polynomial and is defined everywhere

Possible Inflection Points: $x = 0$

6. **Concave Up/Down; Inflection Points:**



$f(x)$ ***is concave down on*** $(-\infty, 0)$
$f(x)$ ***is concave up on*** $(0, \infty)$
Inflection point at $x = 0 \Rightarrow$ ***Point of inflection:*** $(0, 0)$

7. **Asymptotes/End Behavior:**

Vertical Asymptotes	Horizontal Asymptotes/End Behavior
There are no vertical asymptotes since $f(x)$ is continuous everywhere.	$\lim_{x\to\infty}(x^3-3x)=\infty$ $\lim_{x\to-\infty}(x^3-3x)=-\infty$ So, as $x\to\infty, f(x)\to\infty$ and as $x\to-\infty, f(x)\to-\infty$ There are no horizontal asymptotes.

8. **Sketch $f(x)$:** Let's summarize all the information we found in the preceding steps.

   - **Domain:** $(-\infty,\infty)$
   - **Intercepts** at $(0,0)$, $(\sqrt{3},0)$, and $(-\sqrt{3},0)$
   - **Increasing** on $(-\infty,-1)$ and $(1,\infty)$

     Decreasing on $(-1,1)$

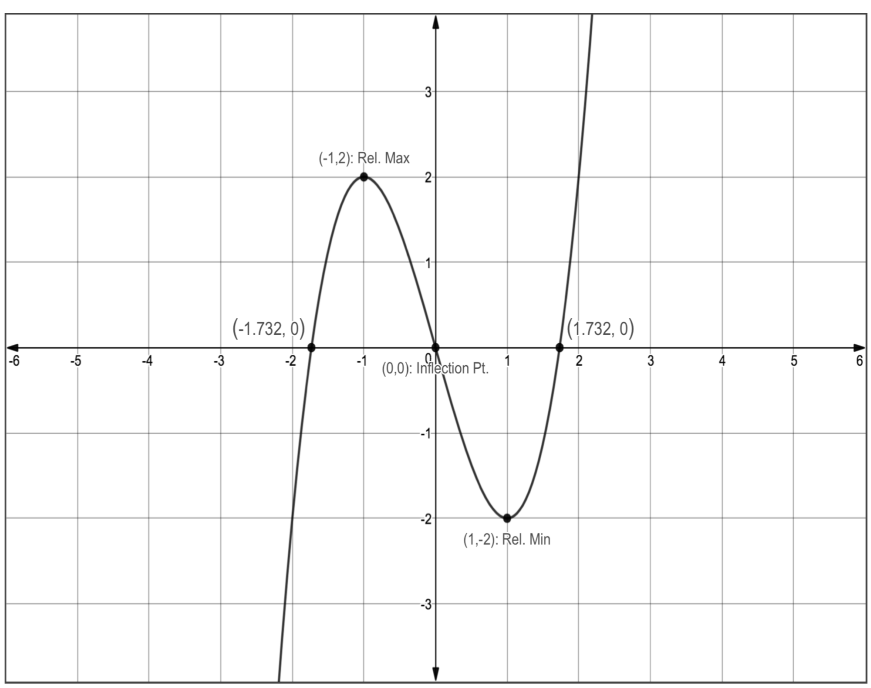
     Concave up on $(0,\infty)$

     Concave down on $(-\infty,0)$
   - **Relative max** at $(-1,2)$

     Relative min at $(1,-2)$

     Inflection point at $(0,0)$
   - **Asymptotes:** There are no horizontal or vertical asymptotes
   - **End Behavior:** As $x\to\infty$, $f(x)\to\infty$
     As $x\to-\infty$, $f(x)\to-\infty$

   Now, using all of this information together, we are able to sketch the graph of $f(x)=x^3-3x$.



Example 6: Given $f(x) = \frac{1}{4}x^4 - x^3$:

- State the domain of $f(x)$.
- State the $x-$ and $y-$intercepts of $f(x)$.
- State the intervals on which the function is increasing/decreasing. Also, state all relative extrema of $f(x)$.
- State the intervals on which the function is concave up/down. Also, state all points of inflection.
- State any vertical or horizontal asymptotes. Also determine the end behavior of $f(x)$.
- Sketch $f(x)$. Label all intercepts, relative extrema, and inflection points.