The Substitution Rule

Calculus I

Modeling Practices in Calculus

1 Indefinite Integrals Review

The set of all antiderivatives of a function is the indefinite integral of that function. If $f'(x)$ is the derivative of $f(x)$, then $f(x)$ is the antiderivative of $f'(x)$.

If

$$\frac{d}{dx}[f(x)] = f'(x)$$

then

$$\int f'(x)dx = f(x) + C$$

Example 1: Evaluate the following.

- $\int x^8 dx$

- $\int (\cos x - \sec x \tan x)\, dx$

- $\int \left(5e^x - \frac{3}{x}\right) dx$

- $\int \left(\frac{1}{\sqrt{1-x^2}} + 4\right) dx$

2 The Substitution Rule

2.1 The Chain Rule Revisited

Let's recall the chain rule: For a composite function, $f(g(x))$,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

So, by the rule for antiderivatives,

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Remember, a key step in the chain rule was to identify the "inner function". So, in the case of $f(g(x))$, we need to identify the function that is $g(x)$. When presented with a "complex" integral we need to evaluate, we can take this same approach of identifying an inner function in order to change integral into a simpler one to evaluate in terms of a new variable.

The Substitution Rule

Given $f'(g(x))g'(x)$, if we let $u = g(x)$, then

$$\begin{aligned} \int f'(g(x))g'(x)dx &= \int f'(u)du \\ &= f(u) + C \\ &= f(g(x)) + C \end{aligned}$$

NOTE: If $u = g(x)$, then $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

Steps for the Substitution Rule

1. Let the new variable, u, be some function of x appearing in the integrand.
2. Take the derivative of u with respect to x.
   - Use this to solve for du in terms of x and dx.
3. Replace all the x-terms in the integral with corresponding u's and du.
4. Evaluate this (simpler) integral in terms of u.
5. Replace all the u's in the final answer back in terms of x. (Don't forget the constant of integration for indefinite integrals.)

NOTE:

- ***Your first choice of u may not work out. If this happens, you should try another choice of u.***
- ***Never let $u = x$.***

2.2 Perfect Substitution

Example 2: Evaluate $\int 2x\sqrt{x^2-9}\ dx$

Let's start by identifying an appropriate choice of u. We should choose u to be a function of x in the integrand that can be considered an "inner function". So, a good choice of u would be:

$$u = x^2 - 9$$

We take the derivative of u with respect to x so that we can find du in terms of x and dx:

$$\begin{aligned} \frac{du}{dx} = \frac{d}{dx}[u] &= \frac{d}{dx}\left[x^2-9\right] = 2x \\ \frac{du}{dx} &= 2x \\ du &= 2xdx \end{aligned}$$

Now, we can replace all the x terms in the integral with the corresponding u terms, evaluate the integral in terms of u, and then put everything back in terms of x.

$$\begin{aligned} \int 2x\sqrt{x^2-9}\ dx &= \int \sqrt{x^2-9}\ 2xdx \leftarrow \textit{rearranging integrand to make the substitution more apparent} \\ &= \int \sqrt{u}\ du \\ &= \int u^{1/2}\ du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}\left(x^2-9\right)^{3/2} + C \leftarrow \textit{Finish by replacing } u \textit{ back in terms of } x \end{aligned}$$

Example 3: Evaluate $\int 4\cos(4x)\ dx$

2.3 Substitution by Introducing a Constant

Example 4: Evaluate $\int x \sin\left(2x^2\right) dx$

Let's start by identifying an appropriate u and finding du:

$$\begin{aligned} u &= 2x^2 \\ \frac{du}{dx} = 4x \Rightarrow du &= 4xdx \end{aligned}$$

We see that du gives us a constant multiple of 4 that is not present in our original integrand. Since we only need the "xdx" portion of du, we can introduce a constant to produce this:

$$\begin{aligned} du &= 4xdx \\ \frac{1}{4}du &= xdx \end{aligned}$$

We are now able to rewrite our integral in terms of u and evaluate:

$$\begin{aligned} \int x \sin\left(2x^2\right) dx &= \int \sin\left(2x^2\right) xdx \\ &= \int \sin(u) \cdot \frac{1}{4}du \\ &= \frac{1}{4}\int \sin(u)du \\ &= \frac{1}{4} \cdot -\cos(u) + C \\ &= -\frac{1}{4}\cos\left(2x^2\right) + C \end{aligned}$$

Example 5: Evaluate $\int e^{-7x} dx$

2.4 Not So Apparent Substitutions

Example 6: Evaluate $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$

At first glance, it may be tempting to let $u = 1 - e^{2x}$. However, if we were to find du with this choice of u, we would obtain $du = -2e^{2x}dx$. This expression (or any variation of it) does not appear in our integrand, so this is not an appropriate choice of u.

Instead, let's first rewrite the expression in the integrand:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$$

From here we can identify another "inner function", e^x, and we can try this as a choice of u:

$$\begin{aligned} u &= e^x \\ \frac{du}{dx} = e^x \Rightarrow du &= e^x dx \end{aligned}$$

Now, we are able to rewrite the integral in terms of u and evaluate:

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx &= \int \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x dx \\ &= \int \frac{1}{\sqrt{1-u^2}}du \\ &= \sin^{-1}(u) + C \\ &= \sin^{-1}(e^x) + C \end{aligned}$$

Example 7: Evaluate $\int \cot(x)dx$

2.5 Substitution Involving Manipulating Variables

Example 8: Evaluate $\int x\sqrt[3]{x+2}\,dx$

An intuitive choice of u is $x+2$, so:

$$\begin{aligned} u &= x+2 \\ \frac{du}{dx} = 1 \Rightarrow du &= dx \end{aligned}$$

Our goal in using substitution to evaluate integrals is to be able to rewrite the entire integrand in terms of u. With our choice of u and du, we can rewrite the $x+2$ and dx parts of the integrand, but we need to also be able to rewrite the x in terms of u. Using algebra, we can use our choice of u to solve for x in terms of u:

$$\begin{gathered} u = x+2 \\ x = u-2 \end{gathered}$$

Now, we can write the given integral in terms of u and evaluate:

$$\begin{aligned} \int x\sqrt[3]{x+2}\,dx &= \int (u-2)\sqrt[3]{u}\,du \\ &= \int (u-2)u^{1/3}\,du \\ &= \int \left(u^{4/3} - 2u^{1/3}\right) du \\ &= \frac{3}{7}u^{7/3} - \frac{3}{2}u^{4/3} + C \\ &= \frac{3}{7}(x+2)^{7/3} - \frac{3}{2}(x+2)^{4/3} + C \end{aligned}$$

Example 9: Evaluate $\int \frac{3x^5}{x^3+5}dx$