Proof of rational root theorem: Suppose $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ with $a_n \neq 0$ and $a_i \in \mathbb{Z}$. Suppose x is a rational root and $x = \frac{p}{q}$. Because all fractions can be in lowest terms, let's suppose it is, i.e. that the greatest common divisor of p and q is 1. So,

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1 \left(\frac{p}{q}\right) + a_0 = 0$$

Multiply thru by q^n to clear qs. This yields

$$0 = a_n p^n + a_{n-1}p^{n-1}q + \cdots a_1 p q^{n-1} + a_0 q^n = 0$$

Isolate the last term. p divides the rest, so $p \mid a_0 q^n$, since $\gcd(p, q) = 1$, $p \mid a_0$. Isolate the first term. q divides the rest, so $q \mid a_n p^n$, since $\gcd(p, q) = 1$, $q \mid a_n$, and p and q are as desired.