23 Recurrence Relations

Proposition 22.3 gives a formula for the sum of the squares of the natural numbers up to n:

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}.$$

How did we derive this formula?

In Exercise 22.16d you were told that a sequence of numbers, $d_0, d_1, d_2, d_3, \ldots$ satisfies the conditions $d_0 = 2$, $d_1 = 5$, and $d_n = 5d_{n-1} - 6d_{n-2}$ and you were asked to prove that $d_n = 2^n + 3^n$. More dramatically, in the same problem, you were asked to prove the following complicated expression for the n^{th} Fibonacci number:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

How did we create these formulas?

In this section we present methods for solving a *recurrence relation*: a formula that specifies how each term of a sequence is produced from earlier terms.

For example, consider a sequence $a_0, a_1, a_2, \ldots$ defined by

$$a_n = 3a_{n-1} + 4a_{n-2}, \quad a_0 = 3, \quad a_1 = 2,$$

We can now compute a_2 in terms of a_0 and a_1, and then a_3 in terms of a_2 and a_1, and so on:

$$\begin{aligned} a_2 &= 3a_1 + 4a_0 = 3 \times 2 + 4 \times 3 = 18 \\ a_3 &= 3a_2 + 4a_1 = 3 \times 18 + 4 \times 2 = 62 \\ a_4 &= 3a_3 + 4a_2 = 3 \times 62 + 4 \times 18 = 258. \end{aligned}$$

Our goal is to have a simple method to convert the recurrence relation into an explicit formula for the n^{th} term of the sequence. In this case, $a_n = 4^n + 2 \cdot (-1)^n$.

First-Order Recurrence Relations

The recurrence relations with which we begin are called *first order* because a_n can be expressed just in terms of the immediate previous element of the sequence, a_{n-1}.

Because the first term of the sequence is a_0, it is not meaningful to speak of the term a_{-1}. Therefore, the recurrence relation holds only for $n \geq 1$. The value of a_0 must be given separately.

The simplest recurrence relation is $a_n = a_{n-1}$. Each term is exactly equal to the one before it, so every term is equal to the initial term, a_0.

Let's try something only slightly more difficult. Consider the recurrence relation $a_n = 2a_{n-1}$. Here, every term is twice as large as the previous term. We also need to give the initial term—say $a_0 = 5$. Then the sequence is $5, 10, 20, 40, 80, 160, \ldots$. It's easy to write down a formula for the n^{th} term of this sequence: $a_n = 5 \times 2^n$.

More generally, if the recurrence relation is

$$a_n = sa_{n-1}$$

then each term is just s times the previous term. Given a_0, the n^{th} term of this sequence is

$$a_n = a_0 s^n.$$

Let's consider a more complicated example. Suppose we define a sequence by

$$a_n = 2a_{n-1} + 3, \quad a_0 = 1.$$

When we calculate the first several terms of this, sequence we find the following values:

$$1, \quad 5, \quad 13, \quad 29, \quad 61, \quad 125, \quad 253, \quad 509, \quad \ldots$$

Because the recurrence relation involves doubling each term, we might suspect that powers of 2 are present in the formula. With this in mind, if we stare at the sequence of values, we might realize that each term is 3 less than a power of 2. We can rewrite the sequence like this:

$$4-3, \quad 8-3, \quad 16-3, \quad 32-3, \quad 64-3, \quad 128-3, \quad 256-3, \quad 512-3, \quad \ldots$$

With this, we obtain $a_n = 4 \cdot 2^n - 3$.

Unfortunately, "stare and hope you recognize" is not a guaranteed procedure. Let's try to analyze this recurrence relation again in a more systematic fashion.

We begin with the recurrence $a_n = 2a_{n-1} + 3$ but leave the initial term a_0 unspecified for the moment. We derive an expression for a_1 in terms of a_0 using the recurrence relation:

$$a_1 = 2a_0 + 3.$$

Next, let's find an expression for a_2. We know that $a_2 = 2a_1 + 3$, and we have an expression for a_1 in terms of a_0. Combining these, we get

$$a_2 = 2a_1 + 3 = 2(2a_0 + 3) + 3 = 4a_0 + 9.$$

Now that we have a_2, we work out an expression for a_3 in terms of a_0:

$$a_3 = 2a_2 + 3 = 2(4a_0 + 9) + 3 = 8a_0 + 21.$$

Here are the first several terms:

$$\begin{aligned} a_0 &= a_0 \\ a_1 &= 2a_0 + 3 \\ a_2 &= 4a_0 + 9 \\ a_3 &= 8a_0 + 21 \\ a_4 &= 16a_0 + 45 \\ a_5 &= 32a_0 + 93 \\ a_6 &= 64a_0 + 189. \end{aligned}$$

One part of this pattern is obvious: a_n can be written as $2^n a_0$ plus something. It's the "plus something" that is still a mystery. We can try staring at the extra terms $0, 3, 9, 21, 45, 93, 189, \ldots$ in the hope of finding a pattern, but we don't want to resort to that. Instead, let's trace out how the term $+189$ was created in a_6. We calculated a_6 from a_5:

$$a_6 = 2a_5 + 3 = 2(32a_0 + 93) + 3$$

so the $+189$ term comes from $2 \times 93 + 3$. Where did the 93 term come from? Let's trace these terms back to the beginning:

$$\begin{aligned} 189 &= 2 \times 93 + 3 \\ &= 2 \times (2 \times 45 + 3) + 3 \\ &= 2 \times (2 \times (2 \times 21 + 3) + 3) + 3 \\ &= 2 \times (2 \times (2 \times (2 \times 9 + 3) + 3) + 3) + 3 \\ &= 2 \times (2 \times (2 \times (2 \times (2 \times 3 + 3) + 3) + 3) + 3) + 3. \end{aligned}$$

Now let's rewrite the last term as follows:

$$\begin{aligned} &2 \times (2 \times (2 \times (2 \times (2 \times 3 + 3) + 3) + 3) + 3) + 3 \\ &= 2^5 \times 3 + 2^4 \times 3 + 2^3 \times 3 + 2^2 \times 3 + 2^1 \times 3 + 2^0 \times 3 \\ &= \left(2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0\right) \times 3 \\ &= \left(2^6 - 1\right) \times 3 = 63 \times 3 = 189 \end{aligned}$$

Based on what we have learned, we predict a_7 to be

$$a_7 = 128a_0 + \left(2^7 - 1\right) \times 3 = 2^7 (a_0 + 3) - 3 = 128a_0 + 381$$

and this is correct.

We are now ready to conjecture the solution to the recurrence relation $a_n = 2a_{n-1} + 3$. It is

$$a_n = (a_0 + 3)\,2^n - 3.$$

Once we have the formula in hand, it is easy to prove it is correct using induction. However, we don't want to go through all that work every time we need to solve a recurrence relation; we want a much simpler method. We seek a ready-made answer to a recurrence relation of the form

$$a_n = sa_{n-1} + t$$

where s and t are given numbers. Based on our experience with the recurrence $a_n = 2a_{n-1} + 3$, we are in a position to guess that the formula for a_n will be of the following form:

$$a_n = (\text{a number}) \times s^n + (\text{a number}).$$

Let's see that this is correct by finding a_1, a_2, etc., in terms of a_0:

$$\begin{aligned}
a_0 &= a_0 \\
a_1 &= sa_0 + t \\
a_2 &= sa_1 + t = s(sa_0 + t) + t = s^2 a_0 + (s+1)t \\
a_3 &= sa_2 + t = s\left(s^2 a_0 + (s+1)t\right) + t = s^3 a_0 + (s^2 + s + 1)t \\
a_4 &= sa_3 + t = s\left(s^3 a_0 + (s^2 + s + 1)t\right) = s^4 a_0 + (s^3 + s^2 + s + 1)t.
\end{aligned}$$

Continuing with this pattern, we see that

$$a_n = s^n a_0 + (s^{n-1} + s^{n-2} + \cdots + s + 1)t.$$

We can simplify this by noticing that $s^{n-1} + s^{n-2} + \cdots + s + 1$ is a geometric series whose sum is

$$\frac{s^n - 1}{s - 1}$$

provided $s \neq 1$ (a case with which we will deal separately). We can now write

$$a_n = a_0 s^n + \left(\frac{s^n - 1}{s - 1}\right) t$$

or, collecting the s^n terms, we have

$$a_n = \left(a_0 + \frac{t}{s-1}\right) s^n - \frac{t}{s-1}. \tag{31}$$

Despite the precise nature of Equation (31), I prefer expressing the answer as in the following result because it is easier to remember and just as useful.

Proposition 23.1 All solutions to the recurrence relation $a_n = sa_{n-1} + t$ where $s \neq 1$ have the form

$$a_n = c_1 s^n + c_2$$

where c_1 and c_2 are specific numbers.

Let's see how to apply Proposition 23.1.

Example 23.2 Solve the recurrence $a_n = 5a_{n-1} + 3$ where $a_0 = 1$.

Solution: We have $a_n = c_1 5^n + c_2$. We need to find c_1 and c_2. Note that

$$\begin{aligned}
a_0 &= 1 = c_1 + c_2 \\
a_1 &= 8 = 5c_1 + c_2.
\end{aligned}$$

Solving these equations, we find $c_1 = \frac{7}{4}$ and $c_2 = -\frac{3}{4}$, and so

$$a_n = \frac{7}{4} \cdot 5^n - \frac{3}{4}.$$

We have a small bit of unfinished business: the case $s = 1$. Fortunately this case is easy. The recurrence relation is of the form

$$a_n = a_{n-1} + t$$

where t is some number. It's easy to write down the first few terms of this sequence and see the result:

$$\begin{aligned}
a_0 &= a_0 \\
a_1 &= a_0 + t \\
a_2 &= a_1 + t = (a_0 + t) + t = a_0 + 2t \\
a_3 &= a_2 + t = (a_0 + 2t) + t = a_0 + 3t \\
a_4 &= a_3 + t = (a_0 + 3t) + t = a_0 + 4t.
\end{aligned}$$

See the pattern? In retrospect, it's pretty obvious.

Proposition 23.3 The solution to the recurrence relation $a_n = a_{n-1} + t$ is

$$a_n = a_0 + nt.$$

Second-Order Recurrence Relations

In a second-order recurrence relation, a_n is specified in terms of a_{n-1} and a_{n-2}. Since the sequence begins with a_0, the recurrence relation is valid for $n \geq 2$. The values of a_0 and a_1 must be given separately.

A second-order recurrence relation gives each term of a sequence in terms of the previous two terms. Consider, for example, the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2}. \tag{32}$$

(This is the recurrence from Exercise 22.16d.) Let us ignore the fact that we already know a solution to this recurrence and do some creative guesswork. A first-order recurrence, $a_n = sa_{n-1}$ has a solution that's just powers of s. Perhaps such a solution is available for Equation (32). We can try $a_n = 5^n$ or perhaps $a_n = 6^n$, but let's hedge our bets and guess a solution of the form $a_n = r^n$ for some number r. We'll substitute this into Equation (32) and hope for the best. Here goes:

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \Rightarrow \quad r^n = 5r^{n-1} - 6r^{n-2}$$

Dividing this through by r^{n-2} gives

$$r^2 = 5r - 6$$

a simple quadratic equation. We can solve this as follows:

$$r^2 = 5r - 6 \quad \Rightarrow \quad 0 = r^2 - 5r + 6 = (r-2)(r-3) \quad \Rightarrow \quad r = 2, 3.$$

This suggests that both 2^n and 3^n are solutions to Equation (32). To see that this is correct, we simply have to check whether 2^n (or 3^n) works in the recurrence. That is, we have to check whether $2^n = 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2}$ (and likewise for 3^n). Here are the proofs:

$$\begin{aligned}
5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} &= 5 \cdot 2^{n-1} - 3 \cdot 2 \cdot 2^{n-2} \\
&= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} \\
&= (5 - 3) \cdot 2^{n-1} = 2^n
\end{aligned}$$

$$\begin{aligned}
5 \cdot 3^{n-1} - 6 \cdot 3^{n-2} &= 5 \cdot 3^{n-1} - 2 \cdot 3 \cdot 3^{n-2} \\
&= 5 \cdot 3^{n-1} - 2 \cdot 3^{n-1} \\
&= (5 - 2) \cdot 3^{n-1} = 3^n.
\end{aligned}$$

We have shown that 2^n and 3^n are solutions to Equation (32). Are there other solutions? Here are two interesting observations.

First, if a_n is a solution to Equation (32), so is ca_n where c is any specific number. To see why, we calculate

$$ca_n = c\,(5a_{n-1} - 6a_{n-2}) = 5(ca_{n-1}) - 6(ca_{n-2}).$$

Since 2^n is a solution to (32), so is $5 \cdot 2^n$.

Second, if a_n and a_n' are both solutions to Equation (32), then so is $a_n + a_n'$. To see why, we calculate:

$$a_n + a_n' = (5a_{n-1} - 6a_{n-2}) + (5a_{n-1}' - 6a_{n-2}') = 5(a_{n-1} + a_{n-1}') - 6(a_{n-2} + a_{n-2}').$$

Since 2^n and 3^n are solutions to Equation (32), so is $2^n + 3^n$.

Based on this analysis, any expression of the form $c_1 2^n + c_2 3^n$ is a solution to Equation (32). Are there any others? The answer is no; let's see why.

We are given that $a_n = 5a_{n-1} - 6a_{n-2}$. Once we have set specific values for a_0 and a_1, $a_2, a_3, a_4, \ldots$ are all determined. If we are given a_0 and a_1, we can set up the equations

$$\begin{aligned}
a_0 &= c_1 2^0 + c_2 3^0 = c_1 + c_2 \\
a_1 &= c_1 2^1 + c_2 3^1 = 2c_1 + 3c_2
\end{aligned}$$

and solve these for c_1, c_2 to get

$$c_1 = 3a_0 - a_1$$
$$c_2 = -2a_0 + a_1.$$

Thus, any solution to Equation (32) can be expressed as

$$a_n = (3a_0 - a_1)\,2^n + (-2a_0 + a_1)\,3^n.$$

Encouraged by this success, we are prepared to tackle the general problem

$$a_n = s_1 a_{n-1} + s_2 a_{n-2} \qquad (33)$$

where s_1 and s_2 are given numbers.

We guess a solution of the form $a_n = r^n$, substitute into Equation (33), and hope for the best:

There is a rough edge in this calculation; since we are dividing by r^{n-2} this analysis is faulty in the case $r = 0$. However, this is not a problem because we check our work in a moment by a different method.

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$
$$r^n = s_1 r^{n-1} + s_2 r^{n-2}$$
$$\Rightarrow \quad r^2 = s_1 r + s_2$$

so the r we seek is a root of the quadratic equation $x^2 - s_1 x - s_2 = 0$. Let's record this as a proposition.

Proposition 23.4 Let s_1, s_2 be given numbers and suppose r is a root of the quadratic equation $x^2 - s_1 x - s_2 = 0$. Then $a_n = r^n$ is a solution to the recurrence relation $a_n = s_1 a_{n-1} + s_2 a_{n-2}$.

Proof. Let r be a root of $x^2 - s_1 x - s_2 = 0$ and observe

$$s_1 r^{n-1} + s_2 r^{n-2} = r^{n-2}(s_1 r + s_2)$$
$$= r^{n-2} r^2 \qquad \text{because } r^2 = s_1 r + s_2$$
$$= r^n.$$

Therefore r^n satisfies the recurrence $a_n = s_1 a_{n-1} + s_2 a_{n-2}$. ■

We're now in a good position to derive the general solution to Equation (33). As we saw with Equation (32), if a_n is a solution to (33), then so is any constant multiple of a_n—that is, $c a_n$. Also, if a_n and a_n' are two solutions to (33), then so is their sum $a_n + a_n'$.

Therefore, if r_1 and r_2 are roots of the polynomial $x^2 - s_1 x - s_2 = 0$, then

$$a_n = c_1 r_1^n + c_2 r_2^n$$

is a solution to Equation (33).

Are these all the possible solutions? The answer is yes in most cases. Let's see what works and where we run into some trouble.

The expression $c_1 r_1^n + c_2 r_2^n$ gives all solutions to (33) provided it can produce a_0 and a_1; if we can choose c_1 and c_2 so that

$$a_0 = c_1 r_1^0 + c_2 r_2^0 = c_1 + c_2$$
$$a_1 = c_1 r_1^1 + c_2 r_2^1 = r_1 c_1 + r_2 c_2$$

then every possible sequence that satisfies (33) is of the form $c_1 r_1^n + c_2 r_2^n$. So all we have to do is solve those equations for c_1 and c_2. When we do, we get this:

$$c_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2} \quad \text{and} \quad c_2 = \frac{-a_1 + a_0 r_1}{r_1 - r_2}.$$

All is well unless $r_1 = r_2$; we'll deal with this difficulty in a moment. First, let's write down what we know so far.

Theorem 23.5 Let s_1, s_2 be numbers and let r_1, r_2 be roots of the equation $x^2 - s_1 x - s_2 = 0$. If $r_1 \neq r_2$, then every solution to the recurrence

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$a_n = c_1 r_1^n + c_2 r_2^n.$$

Example 23.6 Find the solution to the recurrence relation

$$a_n = 3a_{n-1} + 4a_{n-2}, \quad a_0 = 3, \quad a_1 = 2.$$

Solution: Using Theorem 23.5, we find the roots of the quadratic equation $x^2 - 3x - 4 = 0$. This polynomial factors as $x^2 - 3x - 4 = (x-4)(x+1)$ so the roots of the equation are $r_1 = 4$ and $r_2 = -1$. Therefore a_n has the form $a_n = c_1 4^n + c_2(-1)^n$.

To find c_1 and c_2, we note that

$$a_0 = c_1 4^0 + c_2(-1)^0 \quad \Rightarrow \quad 3 = c_1 + c_2$$
$$a_1 = c_1 4^1 + c_2(-1)^1 \quad \Rightarrow \quad 2 = 4c_1 - c_2$$

Solving these gives

$$c_1 = 1 \quad \text{and} \quad c_2 = 2.$$

Therefore $a_n = 4^n + 2 \cdot (-1)^n$.

Example 23.7 The Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Using Theorem 23.5, we solve the quadratic equations $x^2 - x - 1 = 0$ whose roots are $(1 \pm \sqrt{5})/2$. Therefore there is a formula for F_n of the form

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We can work out the values of c_1 and c_2 based on the given values of F_0 and F_1.

Example 23.8 Solve the recurrence relation

$$a_n = 2a_{n-1} - 2a_{n-2} \qquad \text{where } a_0 = 1 \text{ and } a_1 = 3.$$

Solution: The associated quadratic equation is $x^2 - 2x + 2 = 0$, which, by the quadratic formula, has two complex roots: $1 \pm i$. Do not panic. There is nothing in the work we did that required the numbers involved to be real. We now just seek a formula of the form $a_n = c_1(1+i)^n + c_2(1-i)^n$. Examining a_0 and a_1, we have

$$a_0 = 1 = c_1 + c_2$$
$$a_1 = 3 = (1+i)c_1 + (1-i)c_2.$$

Solving these gives $c_1 = \frac{1}{2} - i$ and $c_2 = \frac{1}{2} + i$. Therefore $a_n = (\frac{1}{2} - i)(1+i)^n + (\frac{1}{2} + i)(1-i)^n$.

The Case of the Repeated Root

We now consider the recurrence relations in which the associated polynomial $x^2 - s_1 x - s_2$ has a repeated root. We begin with the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2} \qquad (34)$$

with $a_0 = 1$ and $a_1 = 3$. The first few values of a_n are 1, 3, 8, 20, 48, 112, 256, and 576.

The quadratic equation associated with this recurrence relation is $x^2 - 4x + 4 = 0$, which factors as $(x-2)(x-2)$. So the only root is $r = 2$. We might hope that the formula for a_n

takes the form $a_n = c2^n$, but this is incorrect. Consider the first two terms:

$$a_0 = 1 = c2^0 \quad \text{and} \quad a_1 = 3 = c2^1.$$

The first equation implies $c = 1$ and the second implies $c = \frac{3}{2}$.

We need a new idea. We hope that 2^n is involved in the formula, so we try a different approach. Let us guess a formula of the form

$$a_n = c(n)2^n$$

where we can think of $c(n)$ as a "changing" coefficient. Let's work out the first few values of $c(n)$ based on the values of a_n we already calculated:

$$\begin{aligned}
a_0 &= 1 = c(0)2^0 &&\Rightarrow && c(0) = 1\\
a_1 &= 3 = c(1)2^1 &&\Rightarrow && c(1) = \frac{3}{2}\\
a_2 &= 8 = c(2)2^2 &&\Rightarrow && c(2) = 2\\
a_3 &= 20 = c(3)2^3 &&\Rightarrow && c(3) = \frac{5}{2}\\
a_4 &= 48 = c(4)2^4 &&\Rightarrow && c(4) = 4\\
a_5 &= 112 = c(5)2^5 &&\Rightarrow && c(5) = \frac{7}{2}
\end{aligned}$$

The "changing" coefficient $c(n)$ works out to something simple: $c(n) = 1 + \frac{1}{2}n$. We therefore conjecture that $a_n = \left(1 + \frac{1}{2}n\right)2^n$.

Please note that the solution has the following form: $a_n = c_1 2^n + c_2 n 2^n$. Let's show that all sequences of this form satisfy the recurrence relation in (34):

$$\begin{aligned}
4a_{n-1} - 4a_{n-2} &= 4\left(c_1 2^{n-1} + c_2(n-1)2^{n-1}\right) - 4\left(c_1 2^{n-2} + c_2(n-2)2^{n-2}\right)\\
&= [2c_1 2^n - c_1 2^n] + [2c_2 n 2^n - c_2 n 2^n] + \left[-4 \cdot 2^{n-1} + 8 \cdot 2^{n-2}\right]\\
&= c_1 2^n + c_2 n 2^n + 0 = a_n.
\end{aligned}$$

So every sequence of the form $a_n = c_1 2^n + c_2 n 2^n$ is a solution to Equation (34). Have we found all solutions? Yes we have, because we can choose c_1 and c_2 to match any initial conditions a_0 and a_1; here's how. We solve

$$\begin{aligned}
a_0 &= c_1 2^0 + c_2 \cdot 0 \cdot 2^0\\
a_1 &= c_1 2^1 + c_2 \cdot 1 \cdot 2^1
\end{aligned}$$

which gives

$$c_1 = a_0 \quad \text{and} \quad c_2 = -a_0 + \frac{1}{2}a_1.$$

Since the formula $a_n = 2^n + \frac{1}{2}n2^n$ is of the form $c_1 2^n + c_2 n 2^n$, we know it satisfies the recurrence (34). Substituting $n = 0$ and $n = 1$ in the formula gives the correct values of a_0 and a_1 (namely, 1 and 3), it follows that we have found the solution to Equation (34).

Inspired by this success, we assert and prove the following statement. Notice the requirement that $r \neq 0$; we'll treat the case $r = 0$ as a special case.

Theorem 23.9 Let s_1, s_2 be numbers so that the quadratic equation $x^2 - s_1 x - s_2 = 0$ has exactly one root, $r \neq 0$. Then every solution to the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$a_n = c_1 r^n + c_2 n r^n.$$

Proof. Since the quadratic equation has a single (repeated) root, it must be of the form $(x - r)(x - r) = x^2 - 2rx + r^2$. Thus the recurrence must be $a_n = 2ra_{n-1} - r^2 a_{n-2}$.

To prove the result, we show that a_n satisfies the recurrence and that c_1, c_2 can be chosen so as to produce all possible a_0, a_1.

To see that a_n satisfies the recurrence, we calculate as follows:

$$\begin{aligned}
2ra_{n-1} - r^2 a_{n-2} &= 2r\left(c_1 r^{n-1} + c_2(n-1)r^{n-1}\right) - r^2\left(c_1 r^{n-2} + c_2(n-2)r^{n-2}\right)\\
&= (2c_1 r^n - c_1 r^n) + (2c_2(n-1)r^n - c_2(n-2)r^n)\\
&= c_1 r^n + c_2 n r^n = a_n.
\end{aligned}$$

To see that we can choose c_1, c_2 to produce all possible a_0, a_1, we simply solve

$$\begin{aligned}
a_0 &= c_1 r^0 + c_2 \cdot 0 \cdot r^0 = c_1\\
a_1 &= c_1 r^1 + c_2 \cdot 1 \cdot r = r(c_1 + c_2).
\end{aligned}$$

So long as $r \neq 0$, we can solve these. They yield

$$c_1 = a_0 \quad \text{and} \quad c_2 = \frac{a_0 r - a_1}{r}.$$

■

Finally, in case $r = 0$, the recurrence is simply $a_n = 0$, which means that all terms are zero.

Sequences Generated by Polynomials

We began this section by recalling Proposition 22.3, which gives a formula for the sum of the squares of the natural numbers up to n:

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}.$$

Notice that the formula for the sum of the first n squares is a polynomial expression. In Exercise 22.4b you were asked to show that the sum of the first n cubes is $n^2(n+1)^2/4$, another polynomial expression. Proving these by induction is relatively routine, but how can we figure out the formulas in the first place?

Good news: We now present a simple method for detecting whether a sequences of numbers is generated by a polynomial expression and, if so, for determining the polynomial that created the numbers.

The difference operator Δ should not be confused with the symmetric difference operation, also denoted by Δ. The difference operator converts a sequence of numbers into a new sequence of numbers, whereas the symmetric difference operation takes a pair of sets and returns another set.

The key is the *difference operator*. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. From this sequence we form a new sequence

$$a_1 - a_0, \quad a_2 - a_1, \quad a_3 - a_2, \quad \ldots$$

in which each term is the difference of two consecutive terms of the original sequence. We denote this new sequence as Δa. That is, Δa is the sequence whose n^{th} term is $\Delta a_n = a_{n+1} - a_n$. We call Δ the *difference operator*.

Example 23.10 Let a be the sequence $0, 2, 7, 15, 26, 40, 57, \ldots$. The sequence Δa is $2, 5, 8, 11, 14, 17$. This is easier to see if we write the sequence a on one row and Δa on a second row with Δa_n written between a_n and a_{n+1}.

a:	0		2		7		15		26		40		57
Δa:		2		5		8		11		14		17	

If the sequence a_n is given by a polynomial expression, then we can use that expression to find a formula for Δa. For example, if $a_n = n^3 - 5n + 1$, then

$$\begin{aligned}
\Delta a_n &= a_{n+1} - a_n\\
&= \left[(n+1)^3 - 5(n+1) + 1\right] - \left[n^3 - 5n + 1\right]\\
&= n^3 + 3n^2 + 3n + 1 - 5n - 5 + 1 - n^3 + 5n - 1\\
&= 3n^2 + 3n - 4.
\end{aligned}$$

The *degree* of a polynomial expression is the largest exponent appearing in the expression. For example, $3n^5 - n^2 + 10$ is a degree-5 polynomial in n.

Notice that the difference operator converted a degree-3 polynomial formula, $n^3 - 5n + 1$, into a degree-2 polynomial.

Proposition 23.11 Let a be a sequence of numbers in which a_n is given by a degree-d polynomial in n where $d \geq 1$. Then Δa is a sequence given by a polynomial of degree $d-1$.

Proof. Suppose a_n is given by a polynomial of degree d. That is, we can write

$$a_n = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

where $c_d \neq 0$ and $d \geq 1$. We now calculate Δa_n:

$$\begin{aligned}
\Delta a_n &= a_{n+1} - a_n \\
&= \left[c_d(n+1)^d + c_{d-1}(n+1)^{d-1} + \cdots + c_1(n+1) + c_0\right] \\
&\quad - \left[c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0\right] \\
&= \left[c_d(n+1)^d - c_d n^d\right] + \left[c_{d-1}(n+1)^{d-1} - c_{d-1} n^{d-1}\right] + \cdots \\
&\quad + \left[c_1(n+1) - c_1 n\right] + \left[c_0 - c_0\right].
\end{aligned}$$

Each term on the last line is of the form $c_j(n+1)^j - c_j n^j$. We expand the $(n+1)^j$ term using the Binomial Theorem (Theorem 17.8) to give

$$\begin{aligned}
c_j(n+1)^j - c_j n^j &= c_j\left[n^j + \binom{j}{1} n^{j-1} + \binom{j}{2} n^{j-2} + \cdots + \binom{j}{j} n^0\right] - c_j n^j \\
&= c_j\left[\binom{j}{1} n^{j-1} + \binom{j}{2} n^{j-2} + \cdots + \binom{j}{j}\right].
\end{aligned}$$

Notice that $c_j(n+1)^j - c_j n^j$ is a polynomial of degree $j-1$. Thus, if we look at the full expression for Δa_n, we see that the first term $c_d(n+1)^d - c_d n^d$ is a polynomial of degree $d-1$ (because $c_d \neq 0$) and none of the subsequent terms can cancel the n^{d-1} term because they all have degree less than $d-1$. Therefore Δa_n is given by a polynomial of degree $d-1$. ■

If a is given by a polynomial of degree d, then Δa is given by a polynomial of degree $d-1$. This implies that $\Delta(\Delta a)$ is given by a polynomial of degree $d-2$, and so on. Instead of $\Delta(\Delta a)$, we write $\Delta^2 a$. In general, $\Delta^k a$ is $\Delta(\Delta^{k-1} a)$ and $\Delta^1 a$ is just Δa.

What happens if we apply Δ repeatedly to a polynomially generated sequence? Each subsequent sequence is a polynomial of one lower degree until we reach a polynomial of degree zero—which is just a constant. If we apply Δ one more time, we arrive at the all-zero sequence!

Corollary 23.12 If a sequence a is generated by a polynomial of degree d, then $\Delta^{d+1} a$ is the all-zeros sequence.

Example 23.13 The sequence $0, 2, 7, 15, 26, 40, 57, \ldots$ from Example 23.10 is generated by a polynomial. Repeatedly applying Δ to this sequence gives this:

a:	0		2		7		15		26		40		57
Δa:		2		5		8		11		14		17	
$\Delta^2 a$:			3		3		3		3		3		
$\Delta^3 a$:				0		0		0		0			

Corollary 23.12 tells us that if a_n is given by a polynomial expression, then repeated applications of Δ will reduce this sequence to all zeros. We now seek to prove the converse; that is, if there is a positive integer k such that $\Delta^k a_n$ is the all-zeros sequence, then a_n is given by a polynomial formula. Furthermore, we develop a simple method for deducing the polynomial that generates a_n.

Our first tool is the following simple proposition.

Proposition 23.14 Let a, b, and c be sequences of numbers and let s be a number.

(1) If, for all n, $c_n = a_n + b_n$, then $\Delta c_n = \Delta a_n + \Delta b_n$.
(2) If, for all n, $b_n = s a_n$, then $\Delta b_n = s \Delta a_n$.

For those who have studied linear algebra. If we think of a sequence as a vector (with infinitely many components), then Proposition 23.14 says that Δ is a linear transformation.

This proposition can be written more succinctly as follows: $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$ and $\Delta(s a_n) = s \Delta a_n$.

Proof. Suppose first that for all n, $c_n = a_n + b_n$. Then

$$\begin{aligned}
\Delta c_n &= c_{n+1} - c_n \\
&= (a_{n+1} + b_{n+1}) - (a_n + b_n) \\
&= (a_{n+1} - a_n) + (b_{n+1} - b_n) \\
&= \Delta a_n + \Delta b_n.
\end{aligned}$$

Next, suppose that $b_n = s a_n$. Then

$$\Delta b_n = b_{n+1} - b_n = s a_{n+1} - s a_n = s\,(a_{n+1} - a_n) = s \Delta a_n.$$ ■

The next step is to understand how Δ treats some particular polynomial sequences. We start with a specific example.

Not only can $\binom{n}{3}$ be expressed as a polynomial in n, but the same is true for all $\binom{n}{k}$ (where k is a positive integer). Using Theorem 17.12, when $n \geq k$, write $\binom{n}{k}$ as

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

For the case $0 \leq n < k$, observe that both $\binom{n}{k}$ and the polynomial evaluate to zero. Thus for every positive integer k, $\binom{n}{k}$ can be written as a polynomial of degree k.

Let a be the sequence whose n^{th} term is $a_n = \binom{n}{3}$. For example, $a_5 = \binom{5}{3} = 10$. By Theorem 17.12, we can write

$$a_n = \binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)(n-3)(n-4)\cdots(2)(1)}{(n-3)(n-4)\cdots(2)(1)\cdot 3!} = \frac{1}{6}n(n-1)(n-2)$$

which is a polynomial. This formula is correct, but there is a minor error. The formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ applies only when $0 \leq k \leq n$. The first few terms of the sequence, a_0, a_1, a_2, are $\binom{0}{3}$, $\binom{1}{3}$, and $\binom{2}{3}$. All of these evaluate to zero, but Theorem 17.12 does not apply to them. Fortunately, the polynomial expression $\frac{1}{6}n(n-1)(n-2)$ also evaluates to zero for $n = 0, 1, 2$, so the formula $a_n = \frac{1}{6}n(n-1)(n-2)$ is correct for all values of n.

Now let's calculate Δa_n, $\Delta^2 a_n$, and so on, until we reach the all-zeros sequence (which, by Corollary 23.12, should be by $\Delta^4 a_n$).

a_n:	0		0		0		1		4		10		20		35		56
Δa_n:		0		0		1		3		6		10		15		21	
$\Delta^2 a_n$:			0		1		2		3		4		5		6		
$\Delta^3 a_n$:				1		1		1		1		1		1			
$\Delta^4 a_n$:					0		0		0		0		0				

Please note that every row of this table begins with a zero except for row $\Delta^3 a_n$, which begins with a one.

Since $a_n = \binom{n}{3}$ is a polynomial of degree 3, we know that Δa_n is a polynomial of degree 2. Let's work this out algebraically:

$$\begin{aligned}\Delta a_n = \Delta\binom{n}{3} &= \binom{n+1}{3} - \binom{n}{3} \\ &= \frac{1}{6}(n+1)(n)(n-1) - \frac{1}{6}n(n-1)(n-2) \\ &= \frac{(n^3-n)-(n^3-3n^2+2n)}{6} = \frac{3n^2-3n}{6} \\ &= \frac{1}{2}n(n-1) = \binom{n}{2}.\end{aligned}$$

Having discovered that $\Delta\binom{n}{3} = \binom{n}{2}$, we wonder whether there is an easier way to prove this (there is) and whether this generalizes (it does).

We seek a quick way to prove that $\Delta\binom{n}{3} = \binom{n}{2}$. This can be rewritten $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$, which can be rearranged to $\binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}$. This follows directly from Pascal's Identity (Theorem 17.10).

Seeing that $\Delta\binom{n}{3} = \binom{n}{2}$, it's not a bold leap to guess that $\Delta\binom{n}{4} = \binom{n}{3}$, or in general $\Delta\binom{n}{k} = \binom{n}{k-1}$. The proof is essentially a direct application of Pascal's Identity (with a bit of care in the case $n < k$).

Proposition 23.15 Let k be a positive integer and let $a_n = \binom{n}{k}$ for all $n \ge 0$. Then $\Delta a_n = \binom{n}{k-1}$.

Proof. We need to show that $\Delta\binom{n}{k} = \binom{n}{k-1}$ for all $n \ge 0$. This is equivalent to $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$ which in turn is the same as

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{35}$$

By Pascal's Identity (Theorem 17.10), Equation (35) holds whenever $0 < k < n+1$, so we need only concern ourselves with the case $n+1 \le k$ (i.e., $n \le k-1$).

In the case $n < k-1$, all three terms, $\binom{n+1}{k}$, $\binom{n}{k}$, and $\binom{n}{k-1}$, equal zero, so (35) holds.

In the case $n = k-1$, we have $\binom{n+1}{k} = \binom{k}{k} = 1$, $\binom{n}{k} = \binom{k-1}{k} = 0$, and $\binom{n}{k-1} = \binom{k-1}{k-1} = 1$, and (35) reduces to $1 = 0 + 1$. ∎

Earlier we noted that for $a_n = \binom{n}{3}$, we have that $\Delta^j a_0 = 0$ for all j except $j = 3$, and $\Delta^3 a_0 = 1$. This generalizes. Let k be a positive integer and let $a_n = \binom{n}{k}$. Because a_n is expressible as a degree-k polynomial, $\Delta^{k+1} a_n = 0$ for all n. Using Proposition 23.15, we have that $a_0 = \Delta a_0 = \Delta^2 a_0 = \cdots = \Delta^{k-1} a_0 = 0$ but $\Delta^k a_k = 1$; see Exercise 23.5.

Thus, for the sequence $a_n = \binom{n}{k}$, we know (1) that $\Delta^{k+1} a_n = 0$ for all n, (2) the value of a_0, and (3) the value of $\Delta^j a_0$ for $1 \le j < k$. We claim that these three facts uniquely determine the sequence a_n. Here is a careful statement of that assertion.

Proposition 23.16 Let a and b be sequences of numbers and let k be a positive integer. Suppose that

- $\Delta^k a_n$ and $\Delta^k b_n$ are zero for all n,
- $a_0 = b_0$, and
- $\Delta^j a_0 = \Delta^j b_0$ for all $1 \le j < k$.

Then $a_n = b_n$ for all n.

Proof. The proof is by induction on k.

The basis case is when $k = 1$. In this case we are given that $\Delta a_n = \Delta b_n = 0$ for all n. This means that $a_{n+1} - a_n = 0$ for all n, which implies that $a_{n+1} = a_n$ for all n. In other words, all terms in a_n are identical. Likewise for b_n. Since we also are given that $a_0 = b_0$, the two sequences are the same.

Now suppose (induction hypothesis) that the Proposition has been proved for the case $k = \ell$. We seek to prove the result in the case $k = \ell + 1$. To that end, let a and b be sequences such that

- $\Delta^{\ell+1} a_n = \Delta^{\ell+1} b_n = 0$ for all n,
- $a_0 = b_0$, and
- $\Delta^j a_0 = \Delta^j b_0$ for all $1 \le j < \ell + 1$.

Consider the sequences $a'_n = \Delta a_n$ and $b'_n = \Delta b_n$. By our hypotheses we see that $\Delta^\ell a'_n = \Delta^\ell b'_n = 0$ for all n, $a'_0 = b'_0$, and $\Delta^j a'_0 = \Delta^j b'_0$ for all $1 \le j < \ell$. Therefore, by induction, a' and b' are identical (i.e., $a'_n = b'_n$ for all n).

We now prove that $a_n = b_n$ for all n. Suppose, for the sake of contradiction, that a and b were different sequences. Choose m to be the smallest subscript so that $a_m \ne b_m$. Note that $m \ne 0$ because we are given $a_0 = b_0$; thus $m > 0$. Thus we know $a_{m-1} = b_{m-1}$. We also know that $a'_{m-1} = b'_{m-1}$; here is why:

$$\begin{aligned} a'_{m-1} &= \Delta a_{m-1} = a_m - a_{m-1} \\ = b'_{m-1} &= \Delta b_{m-1} = b_m - b_{m-1} \end{aligned}$$

$$\begin{aligned} a_m - a_{m-1} &= b_m - b_{m-1} \\ a_m - b_m &= a_{m-1} - b_{m-1} = 0 \\ \therefore \quad a_m &= b_m \qquad \Rightarrow\Leftarrow \end{aligned}$$

Thus $a_n = b_n$ for all n. ∎

We are now ready to present our main result about sequences generated by polynomial expressions.

Theorem 23.17 Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The terms a_n can be expressed as polynomial expressions in n if and only if there is a nonnegative integer k such that for all $n \ge 0$ we have $\Delta^{k+1} a_n = 0$. In this case,

$$a_n = a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^k a_0)\binom{n}{k}.$$

Proof. One half of the if-and-only-if statement has already been proved: If a_n is given by a polynomial of degree d, then $\Delta^{d+1} a_n = 0$ for all n (Corollary 23.12).

Suppose now that a is a sequence of numbers and that there is a natural number k such that for all n, $\Delta^{k+1} a_n = 0$. We prove that a_n is given by a polynomial expression by showing that a_n is equal to

$$b_n = a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^k a_0)\binom{n}{k}.$$

To show that $a_n = b_n$ for all n, we apply Proposition 23.16; that is, we need to prove

(1) $\Delta^{k+1} a_n = \Delta^{k+1} b_n = 0$ for all n,
(2) $a_0 = b_0$, and
(3) $\Delta^j a_0 = \Delta^j b_0$ for all $1 \le j \le k$.

We tackle each in turn.

To show (1), note that $\Delta^{k+1} a_n = 0$ for all n by hypothesis. Notice that b_n is a polynomial of degree k, and so $\Delta^{k+1} b_n = 0$ for all n as well (by Corollary 23.12).

It is easy to verify (2) by substituting $n = 0$ into the expression for b_n; every term except the first evaluates to zero, and the first term is $a_0\binom{0}{0} = a_0$.

Finally, we need to prove (3). The notation can become confusing as we calculate $\Delta^j b_n$—there will be too many Δs crawling around the page! To make our work easier to read, we let

$$c_0 = a_0,\ c_1 = \Delta a_0,\ c_2 = \Delta^2 a_0,\ \ldots,\ c_k = \Delta^k a_0$$

and so we can rewrite b_n as

$$b_n = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_k\binom{n}{k}.$$

Now, to calculate $\Delta^j b_n$ we apply Proposition 23.14, Proposition 23.15, and Corollary 23.12:

$$\begin{aligned}\Delta^j b_n &= \Delta^j\left[c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_k\binom{n}{k}\right]\\ &= c_0\Delta^j\binom{n}{0} + c_1\Delta^j\binom{n}{1} + c_2\Delta^j\binom{n}{2} + \cdots + c_k\Delta^j\binom{n}{k}\\ &= 0 + \cdots + 0 + c_j\Delta^j\binom{n}{j} + c_{j+1}\Delta^j\binom{n}{j+1} + \cdots + c_k\Delta^j\binom{n}{k}\\ &= c_j\binom{n}{0} + c_{j+1}\binom{n}{1} + \cdots + c_k\binom{n}{k-j}.\end{aligned}$$

We substitute $n = 0$ into this, which gives

$$\Delta^j b_0 = c_j + 0 + \cdots + 0 = \Delta^j a_0$$

and this completes the proof. ■

Example 23.18 We return to the sequence presented in Examples 23.10 and 23.13: 0, 2, 7, 15, 26, 40, 57, ... We calculated successive differences and found this:

a:	0		2		7		15		26		40		57
Δa:		2		5		8		11		14		17	
$\Delta^2 a$:			3		3		3		3		3		
$\Delta^3 a$:				0		0		0		0			

By Theorem 23.17,

$$a_n = 0\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} = 0 + 2\cdot n + 3\cdot\frac{n(n-1)}{2} = \frac{n(3n+1)}{2}.$$

Example 23.19 Let us derive the following formula from Proposition 22.3:

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}.$$

Let $a_n = 0^2 + 1^2 + \cdots + n^2$. Computing successive differences, we have

a_n:	0		1		5		14		30		55		91		140
Δa_n:		1		4		9		16		25		36		49	
$\Delta^2 a_n$:			3		5		7		9		11		13		
$\Delta^3 a_n$:				2		2		2		2		2			
$\Delta^4 a_n$:					0		0		0		0				

Therefore

$$\begin{aligned}a_n &= 0\binom{n}{0} + 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}\\ &= 0 + n + \frac{3}{2}n(n-1) + \frac{2}{6}n(n-1)(n-2)\\ &= \frac{2n^3 + 3n^2 + n}{6} = \frac{(2n+1)(n+1)(n)}{6}.\end{aligned}$$

Recap

A recurrence relation for a sequence of numbers is an equation that expresses an element of the sequence in terms of earlier elements. We analyzed first-order recurrence relations of the form $a_n = sa_{n-1} + t$ and second-order recurrence relations of the form $a_n = s_1 a_{n-1} + s_2 a_{n-2}$:

- The recurrence $a_n = sa_{n-1} + t$ has the following solution: If $s \neq 1$, then $a_n = c_1 s^t + c_2$ where c_1, c_2 are specific numbers.
- The solution to the recurrence $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ depends on the roots r_1, r_2 of the quadratic equation $x^2 - s_1 x - s_2 = 0$. If $r_1 \neq r_2$, then $a_n = c_1 r_1^n + c_2 r_2^n$ but if $r_1 = r_2 = r$, then $a_n = c_1 r^n + c_2 n r^n$.

We introduced the difference operator, $\Delta a_n = a_{n+1} - a_n$. The sequence of numbers a_n is generated by a polynomial expression of degree d if and only if $\Delta^{d+1} a_n$ is zero for all n. In this case we can write $a_n = a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^d a_0)\binom{n}{d}$.

23 Exercises

23.1. For each of the following recurrence relations, calculate the first six terms of the sequence (that is, a_0 through a_5). You do not need to find a formula for a_n.
- **a.** $a_n = 2a_{n-1} + 2$, $a_0 = 1$.
- **b.** $a_n = a_{n-1} + 3$, $a_0 = 5$.
- **c.** $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
- **d.** $a_n = 3a_{n-1} - 5a_{n-2}$, $a_0 = 0$, $a_2 = 0$.
- **e.** $a_n = a_{n-1} + a_{n-2} + 1$, $a_0 = a_1 = 1$.
- **f.** $a_n = a_{n-1} + n$, $a_0 = 1$.

23.2. Solve each of the following recurrence relations by giving an explicit formula for a_n. For each, please calculate a_9.
- **a.** $a_n = \frac{2}{3}a_{n-1}$, $a_0 = 4$.
- **b.** $a_n = 10a_{n-1}$, $a_0 = 3$.
- **c.** $a_n = -a_{n-1}$, $a_0 = 5$.
- **d.** $a_n = 1.2a_{n-1}$, $a_0 = 0$.
- **e.** $a_n = 3a_{n-1} - 1$, $a_0 = 10$.
- **f.** $a_n = 4 - 2a_{n-1}$, $a_0 = 0$.
- **g.** $a_n = a_{n-1} + 3$, $a_0 = 0$.
- **h.** $a_n = 2a_{n-1} + 2$, $a_0 = 0$.
- **i.** $a_n = 8a_{n-1} - 15a_{n-2}$, $a_0 = 1$, $a_1 = 4$.
- **j.** $a_n = a_{n-1} + 6a_{n-2}$, $a_0 = 4$, $a_1 = 4$.
- **k.** $a_n = 4a_{n-1} - 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$.
- **l.** $a_n = -6a_{n-1} - 9a_{n-2}$, $a_0 = 3$, $a_1 = 6$.
- **m.** $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 5$, $a_1 = 1$.
- **n.** $a_n = -2a_{n-1} - a_{n-2}$, $a_0 = 5$, $a_1 = 1$.
- **o.** $a_n = 2a_{n-1} + 2a_n$, $a_0 = 3$, $a_1 = 3$.
- **p.** $a_n = 2a_{n-1} - 5a_{n-2}$, $a_0 = 2$, $a_1 = 3$.

23.3. Each of the following sequences is generated by a polynomial expression. For each, find the polynomial expression that gives a_n.
- **a.** 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, ...
- **b.** 6, 5, 6, 9, 14, 21, 30, 41, 54, 69, ...
- **c.** 4, 4, 10, 28, 64, 124, 214, 340, 508, 724, ...
- **d.** 5, 16, 41, 116, 301, 680, 1361, 2476, 4181, 6656, ...

23.4. Explain why the notation Δa_n has implicit parentheses $(\Delta a)_n$ and why $\Delta(a_n)$ is not correct.

23.5. Let k be a positive integer and let $a_n = \binom{n}{k}$. Prove that $a_0 = \Delta a_0 = \Delta^2 a_0 = \cdots = \Delta^{k-1} a_0 = 0$ and that $\Delta^k a_0 = 1$.

23.6. Suppose that the sequence a satisfies the recurrence $a_n = a_{n-1} + 12a_{n-2}$ and that $a_0 = 6$ and $a_5 = 4877$. Find an expression for a_n.

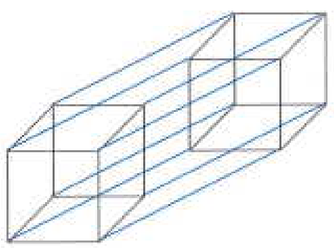
23.7. Find a polynomial formula for $1^4 + 2^4 + 3^4 + \cdots + n^4$.

23.8. Let t be a positive integer. Prove that $1^t + 2^t + 3^t + \cdots + n^t$ can be written as a polynomial expression.

23.9. Some so-called intelligence tests often include problems in which a series of numbers is presented and the subject is required to find the next term of the sequence. For example, the sequence might begin 1, 2, 4, 8. No doubt the examiner is looking for 16 as the next term.

Show how to "outsmart" the intelligence test by finding a polynomial expression (of degree 3) for a_n such that $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8$, but $a_4 = 15$.

23.10. Let s be a real number with $s \neq 0$. Find a sequence a so that $a_n = s\Delta a_n$ and $a_0 = 1$.



23.11. For a natural number n, the n-cube is a figure created by the following recipe. The 0-cube is simply a point. For $n > 0$, we construct an n-cube by taking two disjoint copies of an $(n-1)$-cube and then joining corresponding points in the two cubes by line segments. Thus, a 1-cube is simply a line segment and a 2-cube is a quadrilateral. The figure shows the construction of a 4-cube from two copies of a 3-cube. Note that an n-cube has twice as many points as an $(n-1)$-cube; therefore, an n-cube has 2^n points. The question is, how many line segments does an n-cube have? Let a_n denote the number of line segments in an n-cube. We have $a_0 = 0, a_1 = 1, a_2 = 4, a_3 = 12$, and $a_4 = 32$.

a. Calculate a_5.
b. Find a formula for a_n in terms of a_{n-1}.
c. Find a formula for a_n just in terms of n (and not in terms of a_{n-1}) and use part (b) to prove that your formula is correct.

23.12. Solve the equation $\Delta^2 a_n = -a_n$ with $a_0 = a_1 = 2$.

23.13. Find two different sequences a and b for which $\Delta a_n = \Delta b_n$ for all n.

23.14. The second-order recurrence relations we solved were of the form $a_n = s_1a_{n-1} + s_2a_{n-2}$. In this problem we extend this to relations of the form $a_n = s_1a_{n-1} + s_2a_{n-2} + t$. Typically (but not always) the solution to such a relation is of the form $a_n = c_1r_1^n + c_2r_2^n + c_3$ where c_1, c_2, c_3 are specific numbers, and r_1, r_2 are roots of the associated quadratic equation $x^2 - s_1x - s_2 = 0$. However, if one of these roots is 1, or if the roots are equal to each other, another form of solution is required.

Please solve the following recurrence relations. In the cases where the standard form does not apply, try to work out an appropriate alternative form, but if you get stuck, please consult the Hints (Appendix A).

a. $a_n = 5a_{n-1} - 6a_{n-2} + 2, a_0 = 1, a_1 = 2$.
b. $a_n = 4a_{n-1} + 5a_{n-2} + 4, a_0 = 2, a_1 = 3$.
c. $a_n = 2a_{n-1} + 4a_{n-2} + 6, a_0 = a_1 = 4$.
d. $a_n = 3a_{n-1} - 2a_{n-2} + 5, a_0 = a_1 = 3$.
e. $a_n = 6a_{n-1} - 9a_{n-2} - 2, a_0 = -1, a_1 = 4$.
f. $a_n = 2a_{n-1} - a_{n-2} + 2, a_0 = 4, a_1 = 2$.

23.15. Extrapolate from Theorems 23.5 and 23.9 to solve the following third-order recurrence relations.

a. $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}, a_0 = 8, a_1 = 3$, and $a_2 = 27$.
b. $a_n = 2a_{n-1} + 2a_{n-2} - 4a_{n-3}, a_0 = 11, a_1 = 10$, and $a_2 = 32$.
c. $a_n = -a_{n-1} + 8a_{n-2} + 12a_{n-3}, a_0 = 6, a_1 = 19$, and $a_2 = 25$.
d. $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}, a_0 = 3, a_1 = 2$, and $a_2 = 36$.

23.16. Suppose you wish to generate elements of a recurrence relation using a computer program. It is tempting to write such a program recursively.

For example, consider the recurrence $a_n = 3a_{n-1} - 2a_{n-2}, a_0 = 1, a_1 = 5$. Here is a program to calculate the values a_n:

```
procedure get_term(n)
   if (n < 0)
      print 'Illegal argument'
      exit
   end

   if (n == 0)
      return 1
   end

   if (n == 1)
      return 5
```



```
   end

   return 3*get_term(n-1) - 2*get_term(n-2)
end
```

Although this program is easy to understand, it is extremely inefficient. Explain why.

In particular, let b_n be the number of times this routine is called when it calculates a_n. Find a recurrence—and solve it!—for b_n.

23.17. There are many types of recurrence relations that are of different forms from those presented in this section. Try your hand at finding a formula for a_n for these:

a. $a_n = na_{n-1}, a_0 = 1$.
b. $a_n = a_{n-1}^2, a_0 = 2$.
c. $a_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}, a_0 = 1$.
d. $a_n = na_0 + (n-1)a_1 + (n-2)a_2 + \cdots + 2a_{n-2} + 1a_{n-1}, a_0 = 1$.
e. $a_n = 3.9a_{n-1}(1 - a_{n-1}), a_0 = \frac{1}{2}$.

23.18. The *Catalan numbers* are a sequence defined by the following recurrence relation:

$$c_0 = 1 \quad \text{and} \quad c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}.$$

Please do the following:

a. Calculate the first several Catalan numbers, say up to c_8.
b. Find a formula for c_n.

Part (b) is quite difficult, so here is a bit of magic to get you to an answer. The *On-Line Encyclopedia of Integer Sequences* is a tool into which you can type a list of integers to determine if the sequence has been studied and what is known about the sequence. It is available on the web here: `http://oeis.org/`

c. Use Theorem 23.17 to find a formula for this sequence of numbers: 0, 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210. Please simplify your answer.
d. Use the *On-Line Encyclopedia* to find the name of the sequence in part (c).
e. Finally (just for fun) consider this sequence: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24.... Try to identify this sequence on your own before going to the *On-Line Encyclopedia* for the answer.

Chapter 4 Self Test

1. Prove that the equation $x^2 + 1 = 0$ does not have any real solutions.
2. Prove that there is no integer x such that $x^2 = 2$.
3. Prove that the sum of any four consecutive integers is not divisible by 4.
4. Let a and b be positive integers. Prove: If $a|b$ and $b|a$, then $a = b$.
5. Which of the following sets are well-ordered?
   a. The set of all even integers.
   b. The set of all primes.
   c. $\{-100, -99, -98, \ldots, 98, 99, 100\}$.
   d. $\emptyset$.
   e. The negative integers.
   f. $\{\pi, \pi^2, \pi^3, \pi^4, \ldots\}$ where π is the familiar real number 3.14159....
6. Let n be a positive integer. Prove that
$$1 + 4 + 7 + \cdots + (3n-2) = \frac{3n^2 - n}{2}.$$
7. Let n be a natural number. Prove that
$$0! + 1! + 2! + \cdots + n! \le (n+1)!.$$
8. Suppose $a_0 = 1$ and $a_n = 4a_{n-1} - 1$ when $n \ge 1$. Prove that for all natural numbers n, we have $a_n = (2 \cdot 4^n + 1)/3$.
9. Prove by induction: If $n \in \mathbb{N}$, then $n < 2^n$.