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Examining teachers' understanding of the mathematical learning progression through vertical articulation during Lesson Study

Jennifer Suh · Padmanabhan Seshaiyer

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Abstract This study examines elementary- and middle-grade teachers' understanding of the mathematical learning progression as they participated in a 6-month professional learning project. Teachers participated in a professional development project that consisted of a 1-week summer content-focused institute with school-based follow-up Lesson Study cycles in the fall that focused on the vertical articulation of algebraic concepts across grade levels. The following research examines how a vertical team of teachers from multiple grades designed, taught and learned from the Lesson Study cycle. The video analysis from the research lessons and the teachers' reflections revealed teachers' developing vertical connections and representational fluency from their planning, teaching, observation and debriefs. In addition, Lesson Study afforded teachers opportunities to deepen their understanding of the mathematical learning progression through observation and analysis of students' thinking through a situated school-based professional development experience.

Keywords Learning progression · Professional development · Algebra/algebraic thinking · Vertical articulation, modeling · Problem solving · Representations

Introduction

With the recent release of the Common Core State Standards in the USA (NGACBP and CCSSI 2010), researchers and mathematics educators are looking at the standards to help teachers map out the learning progression that will guide the sequence of mathematical

J. Suh (✉)
College of Education and Human Development, George Mason University, 4400 University Drive, MS: 1E8, Fairfax, VA 22030, USA
e-mail: jsuh4@gmu.edu

P. Seshaiyer
Department of Mathematical Sciences, George Mason University, 4400 University Drive, MS: 3F2, Fairfax, VA 22030, USA
e-mail: pseshaiy@gmu.edu

concepts crucial in building mathematical understanding. Learning progression "extends previous learning while avoiding repetition and large gaps" (Hunt Institute 2012, p. 8) and guides "a path through a conceptual corridor in which there are predictable obstacles and landmarks" (Confery 2012, p. 4). Understanding the learning progression is important for teachers for it serves as the guiding post for analyzing student learning and tailoring their teaching sequence. This essential instructional practice leads to an important practical question: What professional development experiences are necessary for elementary- and middle-grade teachers to develop an understanding of the learning progression? In the US education community, vertical articulation has been coined as the term we use to refer to work involved in helping teachers understand the learning progression across standards. In many cases, learning progressions in a subject domain map out student learning outcomes as they advance through different grade levels and provide standards for formative and summative assessments.

In other educational systems, such as in Japan, elementary school teachers teach all grades from 1 to 6 to allow them to experience how understanding is developed at every level, where the mathematics they are teaching is coming from, and where it is going (Stigler and Hiebert 1999). In many of the pre-service teacher programs in the USA, teacher candidates are required to conduct their student teaching in a primary- and upper-grade placement so that they have experience in planning, teaching and assessing across multiple grade levels. However, once teachers earn their licensure, there is little opportunity for them to observe lessons in other grade levels. Unless, teachers make an intentional commitment to understanding the learning progression, of what comes before and after the grade, the depth of understanding of a subject matter could be limited to the grade level they teach.

This study is aimed at understanding the ways in which the designed professional development can engage teachers in deepening their understanding of students' learning progression of algebraic thinking through Lesson Study at multiple grade levels.

Understanding the mathematics learning progression to build teachers' mathematical knowledge for teaching

Educators and researchers have used a number of definitions and descriptions of what a learning progression entails. Popham (2007) describes a learning progression as a "carefully sequenced set of building blocks that students must master en route to mastering a more distant curricular aim. These building blocks consist of sub-skills and bodies of enabling knowledge" (p. 83). According to Corcoran et al. (2009), learning progression describes students' reasoning as it becomes more sophisticated, and as "…hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time" (p. 37). In the mathematics education research community, the notion of learning trajectories has been important in trying to understand the progression of mathematical concepts and how students' learning progresses and becomes more advanced and sophisticated. Hypothetical learning trajectories include "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon 1995, p. 133). These goals provide direction for teachers as they plan learning activities and are important for teachers to predict the potential reasoning, misconceptions and learning of students. Clements and Samara's work (2004, 2009) takes on a curriculum developers' stance as they use learning trajectories to develop a specific set of sequenced instructional activities. These research-based tasks are designed to promote the child's construction of the skills and concepts of a particular level with a set of sequenced instructional activities hypothesized to be a productive route (Clements and Samaras 2009). Confrey and Maloney (2010) define a learning trajectory, as "a researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation and reflection, toward increasingly complex concepts over time (p. 2)." Understanding the learning trajectory includes components of the learning progression that are important for teachers as they plan instruction, anticipate students' responses, differentiate for diverse learners and assess students' learning.

The work with learning trajectories also supports *vertical teaming* by teachers, for it "allows an exciting chance for teachers to discuss and plan their instruction based on how student learning progresses. An added strength of a learning trajectories approach is that it emphasizes why each teacher, at each grade level along the way, has a critical role to play in each student's mathematical development" (Confery 2012, p. 3). For example, there may be many ways to implement and interpret the Common Core State Standards in the USA or even the district-created teaching and assessment standards. Teachers may understand them, initially, at a surface level, as the big ideas or list of objectives to base their lessons. However, deeper understanding of how these standards are implemented and interpreted in different grade levels is quite a complex task. According to Confery (2012, pp. 7–8), there are five elements that can help unpack a learning trajectory. Teachers need to understand (1) the conceptual principles and the development of the ideas underlying a concept; (2) strategies, representations and misconceptions; (3) meaningful distinctions, definitions and multiple models; (4) coherent structure—recognizing that there is a pattern in the development of mathematical ideas as a concept becomes more complex; and (5) bridging standards—understanding that there might be gaps between standards and knowing what underlying concepts are in between to bridge the gaps between the standards. The complexity of unpacking standards mirrors the complexity of teaching. Understanding the learning progression across grade levels requires the collaboration of teachers through meaningful vertical articulation and professional development.

The deep understanding of the mathematical learning progressions involves important aspects of mathematical knowledge for teaching. Mathematics knowledge for teaching (Hill et al. 2007) includes understanding of general content but also having domain-specific knowledge of students. More specifically, mathematical knowledge for teaching includes practice-based knowledge of "being able to pose meaningful problems, represent ideas carefully with multiple representations, interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems and insights (both predictable and unusual)" (Ball 2003, p. 6). Teachers with mathematical knowledge for teaching must have an extensive and complex set of knowledge and skills that facilitates student learning across the learning progressions so that they can respond to students' responses and move the mathematical agenda forward.

An example of the importance of learning progressions is found in the research on developing algebraic reasoning in the earlier grades through problem solving which requires depth of understanding. Blanton and Kaput (2005, 2008) reported that teachers become better at teaching algebraic reasoning when the teachers' own mathematical knowledge and understanding are increased and their algebra "eyes and ears" allow them to bring out the algebraic reasoning while looking at student work and carefully listening to their discussions and questions. This requires that teachers have an integrated depth of algebra content knowledge to know what to look and listen for in the classroom. It takes a teacher who has a deep and profound understanding of fundamental algebra to explore the foundational concepts for algebraic reasoning through patterning, relations, functions and representations using algebraic symbols and utilizing mathematical models to represent relationships (NCTM 2000). For our research and professional development design, we focused on algebraic thinking by posing problems that represented patterns, function and algebra. Using NCTM's standards (2000, p. 39), we explored algebraic learning progression through rich problems that would expose students to describing recursive patterns, representing mathematics situations with quantitative relationships and by middle grades, "understanding the relationship among tables, graphs, and symbols and to judge the advantages and disadvantages of each way of representing relationships for particular purposes (p. 37)." We wanted teachers to also work with multiple representations of functions, including numeric, graphic, and symbolic, since the representational fluency develops a deeper understanding of functions (Leinhardt et al. 1990; Moschkovich et al. 1993; NRC 1998).

Using vertical school-based Lesson Study teams to focus on learning progressions

Sztajn (2011) reports that mathematics professional development is an emerging research field that needs high quality reports on description of the mathematics professional development and a standard for reporting, including design decisions. The designers and instructors of this project included a university mathematics educator, a mathematician, an elementary mathematics specialist and a middle school Algebra I teacher. We based this project's design on the current research and needs in mathematics education. We considered all the core features of effective mathematics professional development which includes having content as the focus, sustained over time, collective participation of teachers working together focused on issues central to instruction and organized around the instructional materials that teachers use in their classrooms (Garet et al. 2001; Darling-Hammond et al. 2009; Desimone 2009; Little 2003). The content-focused summer institute and the follow-up Lesson Study throughout the academic year focused on engaging teachers in active learning through algebraic problem-solving tasks, exploring pedagogical strategies, utilizing mathematics tools and technology. The content of the professional development promoted algebraic connections aligned to the elementary and middle school curricula and ensured coherence to the standards of learning. Daily activities in the summer institute included modeled lessons using a variety of mathematics tools and technology, as well as in-depth conversation about both the algebra and the pedagogical strategies such as using problem-solving and multiple representations. Our goal was to deepen teachers' algebraic thinking, encourage vertical articulation and develop a productive disposition toward teaching through problem solving. We encouraged collective participation by recruiting school teams that continued into the fall to sustain their learning through a teacher-led professional development model called Lesson Study. These follow-up sessions were designed to provide teachers with continued support in professional learning implementing algebraic content, as well as providing opportunities for vertical articulation between and among grades levels to share ideas and resources, and analyzing student learning.

Our use of Lesson Study is similar to that of the Japanese model (Lewis 2002), which involves engaging teachers in "kyozaikenkyu" research on the teaching and learning materials and assessment. Our teachers engaged in the entire cycle of collaborative planning, studying the curriculum and lesson materials, teaching and observing a research lesson and debriefing after the lesson. In addition, our teachers conducted a second cycle where the lesson was revised and retaught in another teacher's classroom. Unlike, Japan, however, we did not have the infrastructure in place to support Lesson Study, such as built-in professional days, so we built in ways to support this professional development endeavor. The support system included garnering support from school administrators to allow teachers to leave their classroom for a day to participate in the public research lesson event, designing the Lesson Study around a professional development course that was support through a state grant which allowed us to "buy" our teachers release time with substitutes to engage in the public research lesson.

This intentional design of using a school-based Lesson Study as a follow-up to the professional development institute was so that we could examine what teachers were learning *during* and *after* professional development. More specifically, we focused on studying the *coevolution* of participation between classroom practices and professional development, as Kazemi and Hubbard (2008) states, "Teachers' experimentation in classrooms changes the nature of their conversations in PD, and their changing participation in PD leads to new enactments of practice" (p. 432). We were interested in documenting this evolutionary process as we followed teachers as they practiced the *pedagogies of enactment* (Grossman et al. 2009 Grossman and McDonald 2008) where teachers planned for, rehearsed and enacted aspects of practice in the follow-up Lesson Study episodes. It also aligned with a call for moving the learning of teaching closer to practice in teacher education (Gallimore et al. 2009). In addition, Lesson study offered an opportunity to collaboratively research and continuously refine the teaching and learning materials and assessment such as annotated lesson plans, common assessments and other instructional products. The iterative refinement process of teaching and learning materials, according to Hiebert and Morris (2012), contributes to teachers' professional learning, as stated in this statement,

> It is not hard to see how each feature of a lesson plan can improve with each implementation. Information can be gathered on the completeness and clarity of the learning goals and of the rationales, on the effectiveness of the instructional activities, on students' responses, and on the aspects of the lessons for which instructors need more assistance. Those who revise lessons can take advantage of this information to elaborate and refine each feature. This yields continuously improving lessons and increasingly useful knowledge that can guide teachers' implementations of them (Hiebert and Morris 2012, p. 95).

The Lesson Study provides a mechanism of recording and sharing knowledge across local sites for educating teachers as they implement revised lessons and contribute to what Ball et al. (2009) call a *shared library of practice-based materials and structures for collective work*. In addition, Lesson Study allows for collective professional "*noticings*" (Mason 2011; Jacob et al. 2010) during Lesson Study that attends to children's strategies, interpretations and response to student learning. Murata (2011) argues that, *"this implicit and organic* noticing does not happen in artificially replicated professional development settings" (p. 3); therefore, we designed the Lesson Study to provide an authentic setting for this collective and organic noticing to take place. According to Fernandez et al. (2003), Lesson Study also helps teachers develop as teachers as researchers because it allows teachers to put on "researcher-lens" when analyzing observed lessons to focus on the development of meaningful hypotheses/conjectures and think more critically about their curricular materials, learning progression as a result of observing students' responses and students' learning through a rich lesson.

This study used a conjecture-driven research design (Confrey and Lachance 2000) to explore the connections between educational research and the practice of teaching, where our initial conjectures influenced our instructional design and in turn, our instructional design helped us modify and evolve our conjectures. Our initial conjectures (see Fig. 1) were based on the research that teachers needed to engage in sustained professional development focused on the learning progression of mathematics concepts across grade levels. We hypothesized that the learning of the curriculum, materials and assessment were the key mathematical practices that teachers needed to engage in within the context of their teaching. These conjectures led to three areas of opportunity for research: (a) how vertical teams in Lesson Study might afford meaningful articulation of the learning progression in algebraic thinking; (b) how a model like Lesson Study (Lewis 2002) could engage teachers in "kyozaikenkyu"-research on the teaching and learning materials and assessment; (c) how to unpack the learning progression through professional development activities (Fig. 1).

Research questions

To explore how teachers engage in mathematics knowledge for teaching as they collaboratively planned a lesson that focused on the learning progression of algebraic thinking, we asked:

1. In what ways did the vertical teams in Lesson Study provide opportunities for meaningful articulation of the learning progression in algebraic thinking across grade levels?
2. How does an aspect of Lesson Study called "kyozaikenkyu" (research on the teaching and learning materials and assessment) allow for teachers to develop their mathematics knowledge for teaching?

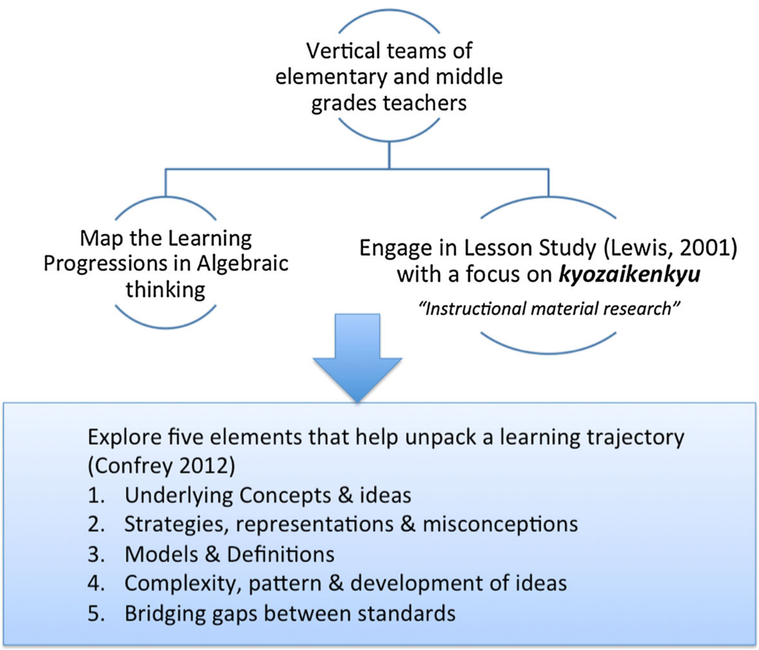


Fig. 1 Initial conjectures for our professional development design and research

3. What are some implications from the proposed professional development model for mathematics educators and professional developers in terms of design decisions?

Methods

In this section, we present the context and content for our professional development and outline our data sources.

Participants

This article focuses on a case study of a vertical team of six teachers and the analysis of a research lesson that was taught in three different grade levels. The lesson study team consisted of a third-grade teacher who was in her third year of teaching, two sixth-grade teachers with more than 5 years of teaching experience, two eighth-grade teachers and a special educator who supported instruction in a fifth-grade class. Among these teachers, the third-grade teacher and one of the eighth-grade teachers were in their first 3 years of teaching while the others ranged from 5 to 8 years. The special education teacher had more than 15 years of teaching experience. All of the participants were new to Lesson Study and were selected into our program because they worked in a Title I school with diverse student populations.

Research context and procedures

This case study was part of a larger project involving thirty-seven elementary- and middle-grade teachers from grades 3–8 who met for a 1-week summer institute and continued to meet as school teams over four to six face-to-face meetings before conducting Lesson Study during the academic year. In order to sustain teachers' professional learning through the academic year, the instructors met with teachers in the fall semester in small vertical teams of five to six multi-grade teachers. We used Lesson Study (Lewis 2002; Lewis et al. 2004) for the follow-up to immerse teachers in vertical teams, collaboratively planning, teaching, observing, debriefing and reflecting both individually and collectively.

In our Lesson Study case study, the multi-grade team of teachers selected a rich algebraic task to teach at multiple grade levels. The team met two sessions to plan their lesson objectives and anticipated student responses. While planning, teachers were asked to focus on these four assessment questions:

1. What does a student at your grade level need to know or be able to do to access this problem?
2. What specific lessons and strategies that you have used in the past will build on this problem?
3. What might be problematic to the students that would require scaffolding in order for them to understand this topic?
4. Develop an assessment task that would be appropriate to your grade level.

As part of the Lesson Study, teachers studied the mathematics content related to the research lesson, planned the lesson, taught the lesson in different grade levels and debriefed after each cycle (see Fig. 2). They decided to start the Lesson Study cycle in a third-grade classroom to see how younger students approached the task. They observed the

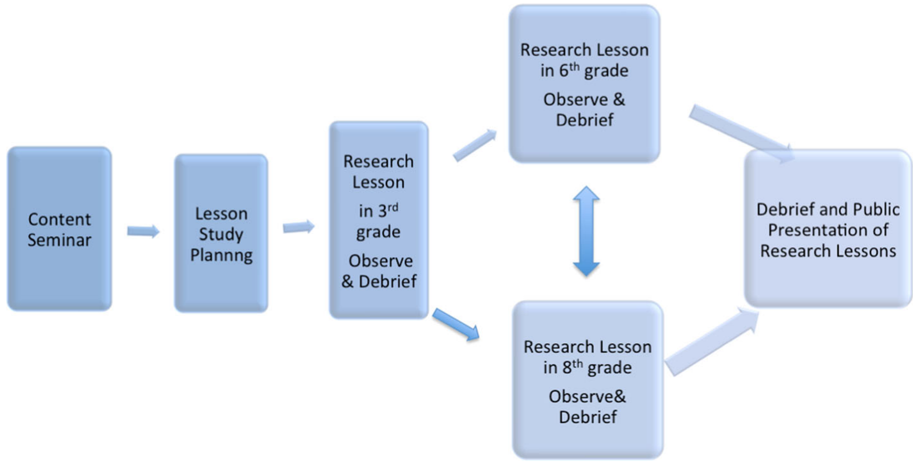


Fig. 2 Sequence of key experiences and process

third-grade host teacher teach the lesson and debriefed the lesson outcome in terms of the teaching moves and the student learning. They collaboratively revised the lesson and geared it up for a sixth-grade class and an eighth-grade classroom. They decided to break up in two teams and observe the lesson and come together as a team. Teachers then met again for the second cycle of the teaching in the upper-grade classrooms and debriefed the lesson for the final cycle together as a group. Finally, they combined their learning from the research lessons from multiple grades to present at the final Lesson Study Conference.

There were a number of goals for these follow-up sessions: to provide teachers with continued support in learning and implementing algebraic content; to supply materials and strategies to the participants; to provide opportunities for vertical articulation between and among grades levels to share ideas/resources; and to analyze student learning and work samples and share their collective "noticings."

Data sources

The data sources included video clips from the research lessons, student work, teacher reflections from the Lesson Study and researchers' memos.

Teacher reflections

For the Lesson Study Reflections, we asked teachers to reflect on the Lesson Study in a journal, focusing on the process of developing and refining a research lesson, creating assessments items and analyzing students' learning. The formal reflection assignment included teachers' evaluation of instructional strategies that promoted algebraic thinking through problem solving, teachers' analysis of student thinking and what was learned from the process of collaboratively planning, teaching, observing and debriefing with colleagues.

Videoclips, memos and artifacts from the Lesson Study

The preplanning, teaching and debrief sessions were captured on video, along with researchers' memos. During the Lesson Study planning cycle, there were at least three opportunities to observe the participants as they worked in small groups to plan, implement and reflect on the research lessons. The artifacts collected from the Lesson Study cycle included the planning agendas, actual lesson plans, student work samples, the analysis of student work and reflections. Each of these factors contributed to compiling a comprehensive picture of teachers' learning experiences.

Data analysis

This project used a qualitative approach using a conjecture-driven model described by Confrey and Lachance (2000) as "a means to reconceptualize the ways in which to approach both the content and the pedagogy of a set of mathematical topics" (p. 237). The conjecture-driven research method allowed us to explore the *content*, the understanding of the learning progression of algebraic thinking and the *pedagogy*, how providing teachers with an environment and experiential activities for collaboration in vertical team during Lesson Study would lead to enhanced understanding of the learning progression. We used Confery (2012) five elements (see Fig. 1) to analyze the unpacking of the learning trajectory during the Lesson Study. This included categorizing the data according to how teachers engaged in dialogue about the (1) underlying concepts and ideas; (2) strategies, representations and misconceptions; (3) models and definitions; (4) complexity, pattern and development of ideas; and (5) bridging gaps between standards. We gathered and analyzed data through the summer institute and the Lesson Study cycle to test our theory. The data analysis was aimed at answering the research questions and identifying themes and categories (Miles and Huberman 1994). Through successive readings of the data, we used the constant comparative method to modify and further develop the coding categories (Bogdan and Biklen 2003).

Analysis of reflections, lesson plans and researchers' memos

To begin analyzing the themes, we used the document analysis technique using teachers' reflections from the Lesson Study, the group lesson plan and the researchers' memos. To display and organize the collected data, we systematically analyzed data using Nvivo, a qualitative software that codes for categories in such a way that draws emerging themes (Miles and Huberman 1994). To verify and compare recurring themes and categories, the research team worked individually on coding the documents before comparing preliminary codes in order to agree upon recurring themes from the reflections. Analyzing additional data collected from teachers' notes from the Lesson Study planning meetings and observations, as well as, analysis of students' work provided descriptive information that linked teachers' reflective comments to actual teaching practices and the implementation process. The ideas, which emerged from the reflections, were categorized into themes and cross-checked with teachers' comments during the videotaped class sessions and the researchers' memos until a set of common themes emerged. To analyze the vertical articulation during the planning and teaching of the research lessons across grade levels, we analyzed the video clips from the planning meeting, the research lessons and the debrief to create a comprehensive picture of teachers' professional learning experiences from the Lesson Study professional development process.

Results

In the results section, we present the analysis of a Lesson Study focused on algebraic thinking called the "Ichiro" problem. The Lesson Study group modified the task that we engaged the teachers in during our summer institute called the "Ichiro" problem, which is a TIMSS research lesson from Japan (Lesson Lab 2004). They chose the problem because it was an accessible problem at multi-grades as it had a progression of solution strategies from simple recording of patterns, to representing the mathematical situation using tables, graphs and number symbols and finally, function-based approach showing linear relationships and solving for inequalities. The analysis of the data sources resulted in three major recurring themes that informed our initial conjectures.

Unpacking the learning progression of algebra

For the first research question, in what ways did the vertical teams in Lesson Study provide opportunities for meaningful articulation of the learning progression in algebraic thinking across grade levels, we found that planning in vertical teams allowed for natural dialogue around learning progressions in algebra and developed teachers' content knowledge, particularly how they selected and modified problems and anticipated students' misconceptions strategies and representations.

The Lesson Study team consisted of third-grade, sixth-grade and eighth-grade teachers, which allowed for rich and natural vertical articulation to take place about how algebraic thinking develops through the grades. This multi-age model also allowed the teachers to map out students' learning progression by identifying grade-appropriate objectives. The Lesson Study team formed with teachers from multi-grades decided to start the first iteration of the lesson in the third-grade classroom so that they would benefit from observing what third graders were capable of doing with the chosen problem. Based on the mathematics standards for the grade levels, each of the three research lesson cycles had a slightly different objective (see Table 1). However, one common objective was to see how students described patterns and expressed relationships through multiple representations like pictures, tables, graphs, number sentences and symbols. For each lesson, teachers provided manipulatives, blank t-charts, graph paper, blank paper and other conceptual supports prepared in the preplanning stages.

The vertical nature of the team make-up allowed for natural dialogue around the curricular objectives that each teacher was responsible for at their grade levels. An excerpt from the planning discussion from the three research lessons indicated a slightly different curricular objective and allowed for teachers to bridge the gaps between standards.

> In the third grade, this problem has the potential to develop algebraic thinking as students look for a pattern as the value of the coins continue to decrease. They may use repeated subtractions as a method of showing the pattern of change. –Third grade host teacher

> In this six grade lesson, we want to see how students organize their information and work through this multistep problem and if they use a table or graph to illustrate the change. -Sixth grade host teacher

> The eighth graders have been working with variables all week, I am curious to see if some groups can come up with an algebraic expression to model the problem. -Eighth grade host teacher.

Table 1 Modified "Ichiro problems" from the research lessons

Original Ichiro problem, a TIMSS research lesson from Japan (Lesson Lab, 2004)

It has been 1 month since Ichiro's mother entered the hospital. He has decided to give a prayer with his small brother at a local temple every morning so that she will be well soon. There are 18 ten yen coins in Ichiro's wallet and just 22 five yen coins in his younger brother's wallet. They decided to place one coin from each of them in the offertory box each morning and continue the prayer until either wallet becomes empty. One day they looked into their wallets and found the brother's amount was bigger than Ichiro's. How many days since they started prayer?

NCTM Process Standards: Problem solving, Connection, Communication, Reasoning and Proof, and Representations

Research Lesson Case Study #1: Piggy bank-3rd grade Dan and Nick want to buy their mother a birthday present next month. Dan has been saving dimes and Nick has been saving nickels. Dan has saved 18 dimes and Nick has saved 22 nickels. The brothers agree to take a coin out of their wallet each day and put it in a piggy bank for their mother's birthday present. One day, when they looked into each other's wallets, they saw that Nick had more money than Dan. When this happened, how many days had they been saving for their mother's gift?	*3rd grade Lesson Objectives* The student will recognize and describe a variety of **patterns** formed using numbers, tables and pictures, and extend the patterns, using the same or different forms. The student will solve single-step and multistep problems involving the sum or difference.

Research Lesson: Aleah's spending-6th grade Lesson Study

Lesson Objectives

The student will describe the relationship found in a **number pattern and express the relationship** using tables, graphs, number sentences and symbols.

Students will solve **multistep practical problems involving linear equations.**

Research Lesson: Wishing for a puppy-8th grade Lesson Study

Lesson Objectives

Students will use **problem-solving methods of inequalities by comparing** and applying the characteristics of **solving a simple linear equation.**

Students will **represent linear relationships** with tables, graphs, rules, and words and make connections between any two representations (tables, graphs, words, and rules) of a given relationship.

The dialogue during the planning meeting led to brainstorming important mathematical strategies in developing algebraic reasoning in the K-8 curriculum, such as analyzing patterns, function and change, solving multistep problems, representing linear relationships and understanding linear inequalities using multiple representations such tables, graphs, rules, and words. While the third-grade teacher's lesson focused on analyzing a pattern of change, the preplanning session also provided middle-grade teachers an opportunity to share possible student responses. The following excerpt highlights teachers' discussion related to possible student difficulties with identifying the variable.

> Ray: So the kids need to know that there is an unknown and they need to know how to identify it.

> Sam: Right, they need to know the variable –be able to accurately identify the variable. It's like when they are working on a word problem and you are trying to find the cost of the orange, they let x equal the number of oranges instead of the cost of the orange.

> Ray: It makes a big difference, like for this problem. It's like the difference between the number of coins versus the amount like the value of the coins and the number of

> days. So that is where we are going. I always start with having the kids find the variables and how the other parts of the word problem relate to the variable, write the equation or inequality and solve and then write me a statement that answers the question.
>
> Cindy: Yeah, so like you said Ray, they might be confused about how to relate the information in the problem and defining the variable to the amount-the value, the number of days or the number of coins.

By having teachers from multiple grade levels plan the lesson, teachers unpacked the standards that aligned to the problem and discussed in detail the prior learning building blocks and the connections to related concepts, which helped them see the pattern in the development of ideas.

Engaging in pedagogical dimensions of addressing the learning progressions

For the second research question, how does an aspect of Lesson Study called 'kyo-zaikenkyu' (research on the teaching and learning materials and assessment) allow for teachers to develop their mathematics knowledge for teaching, we found that it made an impact on teachers' pedagogical knowledge in relation to their modeling strategies and assessment of students' learning progressions.

Modeling through multiple representations and strategies at different grade levels

One common theme among the lessons was the idea of eliciting multiple representations and evaluating efficient uses of representations to develop conceptual understanding and make connections among representations. In the third-grade lesson, preplanning discussions focused on how teachers might expose students to multiple strategies (use of concrete manipulatives: coins, pictures, table, graph) to solve the problem while giving students choice. Evidence from the jointly planned research lessons indicated that teachers were interested in having students compare their strategies and used questioning as a way to elicit students to look for patterns. An excerpt from their teaching notes also indicated that an essential understanding that teachers wanted for students to gain was how to organize information from a given problem.

> Pivotal questions for students: What are some similarities and differences among our strategies? How can you describe your strategy in one sentence? What is happening each day? How will you organize your coins? How will you keep track of how much money Nick and Dan have? What tool could you use to help you solve the problem? Excerpt from 3rd grade research lesson

Through these intentional teaching moves, the Lesson Study team focused on how students navigated among the different representations. By having the tools and representations available, yet not proscribing a method, students were given the choice of using an approach that made most sense to them. This also allowed for teachers to analyze the most common approaches among students and how they interacted with the mathematics. The following is an excerpt from the third-grade teacher's reflection who described students' interaction with the problem and their problem-solving strategies,

> I had two boys come up to the front of the room and pretend they were the brothers who were saving their money to buy their mom a birthday present. My students were

> very interested to watch their peers "act it out". One of my students, a very articulate little girl, created her own table. Her table was very close to what I was hoping to find so I had her present it to our class (See Fig. 3). Working off of her table, I had my class look for patterns on the white board where she was using subtraction. Together we established that the amount of money in Nick and Dan's piggy banks were decreasing. When we "discovered the pattern" I observed that several light bulbs were steadily going off in my students' heads.

Having students work with the 'five representations' allowed the class to name the different strategies: using numbers and symbols, manipulatives, words, graphs and tables. The teacher and the observers shared their *noticings* that students had access to the problem with the physical manipulatives of the coins but when they started putting the coins in their "piggy bank," it was apparent that they had difficulty keeping track. They counted the coins that went in the bank but could not keep track of the amount each of the characters had put in without the help of a recording sheet or a table, which a girl in the class started generating. By working in collaborative groups, others in the group observed and started to mimic her method (See Fig. 3).

> The students in these groups instinctively wanted to draw a picture or create a table, which a lot of them did, but then they could not translate that into words or a number sentence. We also felt that the use of dry erase boards presented a problem because students would erase their work and start over rather than build on their previous mistakes. Teacher collaborator and observer at the third grade research lesson

The idea of accessibility through multiple representations came up in the sixth-grade and the eighth-grade lessons. The sixth-grade teacher noted how the students took the problem and acted it out while one student kept track of the mathematics:

> Another group of students used a concrete model along with a table and tally marks to record their thinking. The three boys each took on a designated role in the problem. Shequeem was Aleah's brother and Raul played Aleah while Daniel was in charge of keeping track of how much money had been spent with tally marks. This team of boys really worked together to both understand and solve the problem by using many representations—verbal, concrete, and with a table. -Sixth grade teacher

Focusing on multiple approaches allowed students greater access to the problem and in turn, more success with the problem. Additionally, teachers promoted the idea of organizing information and keeping track of the changes as an important part of solving the algebraic problems and analyzing the recursive pattern.

An interesting theme that emerged from the collective debrief was the analysis of sophistication of strategies. This multi-grade makeup of grades 3–8 teachers pushed the teams to think across the learning progression and the algebraic vertical connections. In the planning of the lesson, teachers anticipated that students would most likely generate concrete, pictorial, numeric and tabular approaches which were considered to be more accessible for early grades and graphical, verbal and algebraic approaches to be more sophisticated. The team also considered presenting a graphical approach if it was not generated by the students, so that students would see the rate of change and be encourage to interpret the graph based on the context of the problem. In the sixth-grade lesson called *Aleah's Spending*, the teachers wanted to focus on the multi-representational aspect of algebra. The research lesson was taught in a low performing mathematics class and the teacher who led the lesson thought that many of her students needed concrete

Fig. 3 Students engaged in manipulating coins and drawing a table

manipulatives. They decided that they would have $10 and $5 bills available for manipulation. The sixth-grade teacher did not anticipate any of the students creating a graph but did decide to have graph paper available just in case. During the observation of the lesson, the other teacher-observers noted how students negotiated meaning among team members and how they worked together to make sense of the problem. As indicated by the host teacher's reflection, which she shared during the debrief, many responses and strategies surprised her. She stated,

> Timeia originally started working with a group that used concrete models to solve the problem, but she quickly moved off to work on her own and began recording repeated subtraction from both Aleah's bank account and her brother's. This was one of the strategies we thought that students may use in our lesson plan; unfortunately Timeia did not get to finish showing her work for this strategy, but it is clear that she sees a repeated pattern occurring in the problem (see Fig. 4).

One exciting mathematical happening was that one of the students that the host teacher anticipated having the most trouble with the problem actually gravitated toward graphing, which the team of teachers perceived as a more sophisticated and unfamiliar approach. She stated her surprise in her reflection:

> Lastly, one of my students who I anticipated having trouble with the task surprised me when he immediately went to graphing the two situations (see Fig. 5)—Aleah's money and her brother's money. While it took most of his time to set up the scale and the data points did not match up to the days, Thomas could quickly see a pattern of the decreasing values of money in their bank accounts. Seeing Thomas gravitate toward creating a graph was a topic of our conversation during our debriefing session. One of my colleagues even suggested having him graph his multiplication facts; something that he normally struggles with. Maybe a graph is just more sophisticated to him and something new that he will grasp on to.

The Lesson Study team intentionally planned for the students to solve the problem in their most natural way, then to give them opportunities to sit side-by-side with a partner with a venn diagram to compare their approaches. This built-in activity allowed for rich discussion of comparison and connection between the representations. For example, in Fig. 6, one student compared the quantities by crossing off the same number of coins for each of the brothers while another student kept track of the remaining balance using a table.

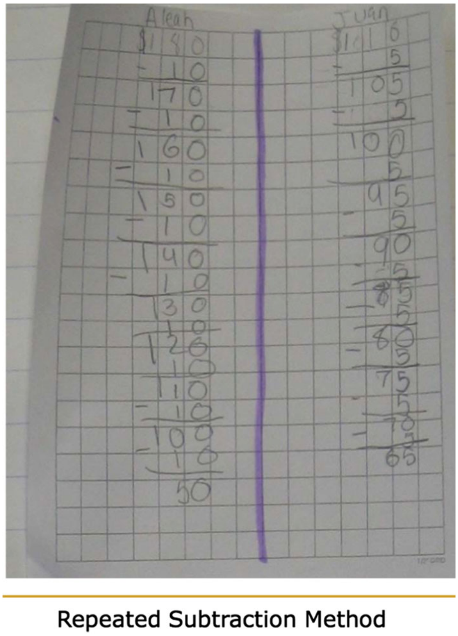


Fig. 4 Using a tabular approach with a repeated subtraction method

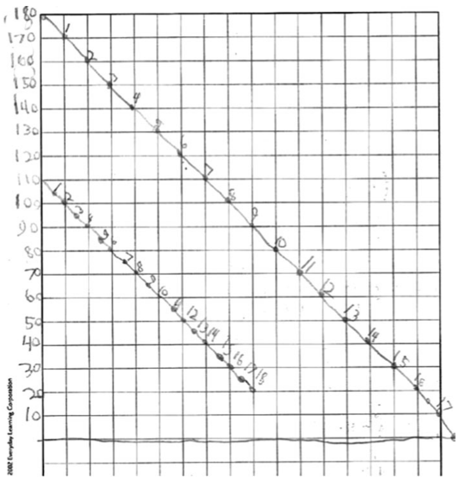


Fig. 5 Graphical approach with error but showing initial ideas about the pattern of change

> One of the focal points of our lesson was to have students look at different strategies to solve the problem and also think about the similarities and differences among their solution paths. We had students use many of the multiple representations that we focused on in our class this summer—concrete, verbal, pictorial, graphically, and symbolic. -Fifth grade teacher special educator

As teachers observed the host teacher teach the first rendition of their collaboratively planned lesson, they made connections to how the lesson would be revised for their respective grades. In doing so, they discussed the vertical algebraic connections stating

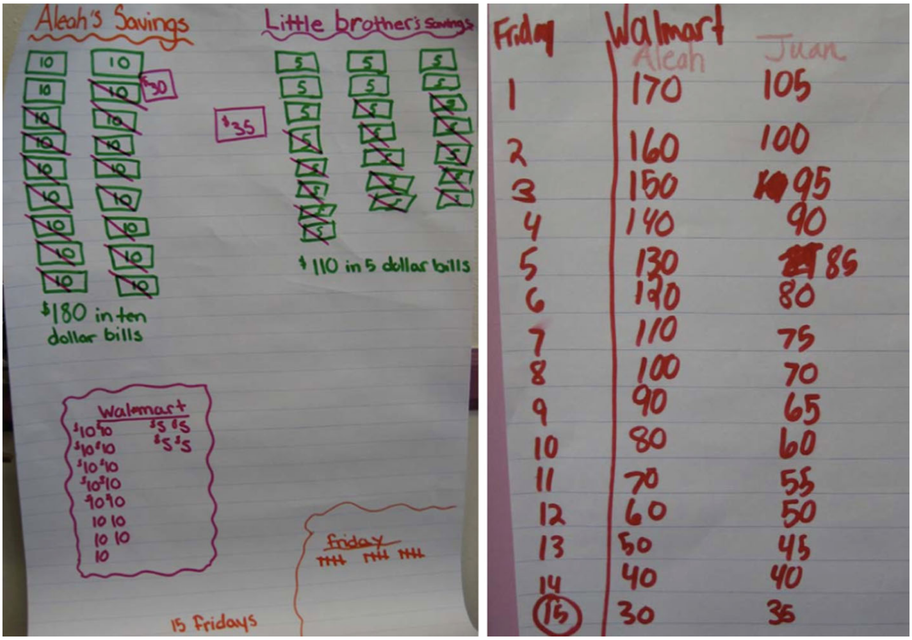
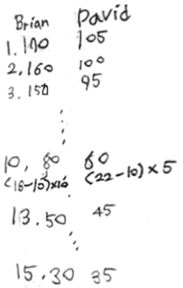


Fig. 6 Sixth-grade students worked individually then comparing strategies

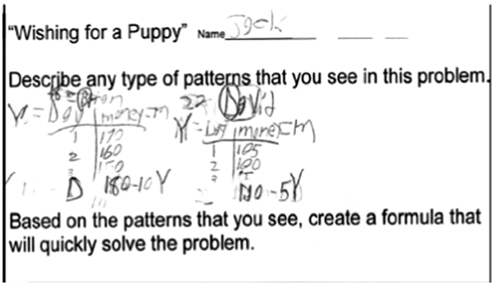
where in the continuum they would find most of their students in their algebraic reasoning with this particular problem. For example, teachers observed how most of the students in the earliest stages of the learning progression represented their thinking through a *repeated subtraction* method or using a *comparing quantity approach* to show the decreasing values. They also had difficulty fully understanding the word problem and setting up mathematical statements for the relationship that was being expressed. The Lesson Study team discussed how the majority of students used the most common two methods we had seen in younger classes—a chart and a drawing. This idea became a critical discussion point. Can students become so fixated by a pictorial or concrete representation that it prevents them from looking for more efficient strategies? Or do most learners need entry into a problem using a concrete, pictorial or tabular approach to make sense of the problem before advancing to an algebraic equation? What is the teachers' role and how do teachers use questioning to advance students' thinking when they are "stuck"? How can we use classroom discourse and representations as a means of moving student from inefficient to sophisticated strategies? The host teacher also made some "horizontal connections" as she related this problem to other algebraic problems with linear equations called the *cell phone plan problem* where students find the best plan using similar strategies as this problem.

Learning about assessing progressions of students' mathematical ideas

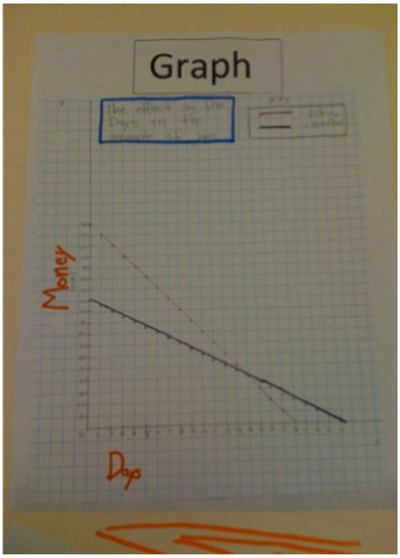
During the institute, teachers wrote reflections to think about their own solution strategies, and during the Lesson Study, they wrote reflection to analyze their students' strategies. The reflection assignment from the Lesson Study served two purposes. In one way, it pushed teachers to analyze how different students approached the problem but for us as designers



Student A



Student B



Daniel's Approach

Daniel approached the problem by using a double-sided table, which showed the decrease in money for both of the brothers. He kept subtracting until he found that David had more money and then counted how many days it had been. Since he also finished quickly, I encouraged him to try and solve the problem in a different way to verify his work. I provided other materials that I let the students know about before they began, such as colored squares and graphing paper, to help them in their discovery. Daniel chose to try a graphical representation as Tiara did. Both these students were in different classes from each other, so it was really nice to see how they came up with the same type of graph. He, too, found the point of intersection was the day at which they had the same amount of money and that the 15th day David had more. Daniel was able formally show his algebraic thinking through both his table and his graph, however had some difficulty verbalizing his findings. But overall, he shows that he is developing skills to think in abstract ways.

Fig. 7 Teacher's analysis of eighth-grade students' thinking and use of representations

and instructors, this served as a way to assess teachers' competence in analyzing multiple representations. The teachers who observed the lesson were impressed by some of the strategies generated by the students.

In Fig. 7 below, the eighth-grade teacher discussed how *Student A* found an efficient way to create the table (see Fig. 7). Instead of listing all the amount decreased each day, he decided to use a number sentence to skip down 10 days using $(18 - 10) \times 10 = 80$ and $(22 - 10) \times 5 = 60$ and used the notation (…) to show that the pattern continued. *Student B* used y as the days and created $180 - 10y$ for Brian and $105 - 5y$ for David, which eventually led to the discussion of solving for $180 - 10d = 110 - 5d$. This excerpt was selected from a collection of Lesson Study reflections because it demonstrated how the teacher's ability to analyze among different representations was deepened by the research lesson. He compared the tabular approach, algebraic expressions and the graphical representations created by his student, Daniel, in the excerpt below (See Fig. 7).

This teacher's excerpt above shows evidence of the depth in which he examined his student's solution strategies, use of representations, sophistication of ideas and justification. The teacher had solved this rich task during the summer institute and had collaborated with other teachers to anticipate multiple strategies and sophistication of methods, which provided the specialized mathematical knowledge to perform this analysis. As a result, the teacher was prepared to analyze students' work and marked the learning progression displayed in his classroom and determined where he might go with his students to push their mathematical thinking forward.

Opportunities to think beyond one's grade level through vertical teaming

Finally, to address the third question, what are some implications from the proposed professional development model for mathematics educators and professional developers in terms of design decisions, we reflected on key activities that participants referred to as pivotal points in the experience that led to new ways of thinking about teaching for algebraic connections across multiple grade bands.

The conjecture driving our study was that the proposed design of the professional development with vertical Lesson Study team would help teachers understand the learning progression in the algebraic lesson. The design of the vertical Lesson Study team pushed teachers to think beyond their grade level. We made some specific design decisions that were instrumental in reaching our goal, which was to help teachers see how algebraic reasoning can be used all throughout the elementary and middle grades' curriculum. Critical to this goal was the design decision to have teachers from grades 3–8 in one professional development project working together through the problems and the Lesson Study. The vertical articulation occurred naturally due to this design element and having Lesson Study take place in multiple grade levels allowed teachers to see the learning progression and curricular progression first hand in a real classroom with real students. This was the most powerful experience for these teachers and a common theme in all of their reflections.

From the analysis of teachers' reflection, we categorized critical incidents or specific activities that prompted relearning, disequilibrium and awareness. We categorized these experiences as the *catalysts for change* in our teachers because these activities led them to rethink and reflect on their instructional practices for mathematics teaching and learning and lead them through a pathway to improving instruction (see Fig. 8). They had to experience first hand what it was to be a learner during Phase 1 of the summer institute and then to implement the lesson and reflect on how students reacted to a new approach to teaching in subsequent phases. As our teacher-participants immersed themselves in the ongoing key activities, each took on multiple roles: as a mathematics learner, curriculum planner, teacher, observer and researcher.

This significant rethinking and reevaluation took place as the teachers performed and then reflected on tasks during the content-focused institute and the Lesson Study. Figure 8 called the *Catalysts for Change* is our framework for marking the distinct phases of the professional development process and experiences that contributed to understanding the learning progression. The first phase of the content-focused institute was associated with the studying and relearning of algebra through multiple representations with vertical connections. The next phases, the Lesson Study included *kyozaikenkyu*, the research and planning of the lesson, which provided opportunities for teachers to translate what they learned into teaching. Teachers from multiple grades engaged in vertical articulation of the scope and sequence between the curricular connections during the planning and implementation process. Phase 3 and 4, the teaching, observing and debriefing process of the Lesson Study provided teachers an opportunity to assess students' learning progression in a

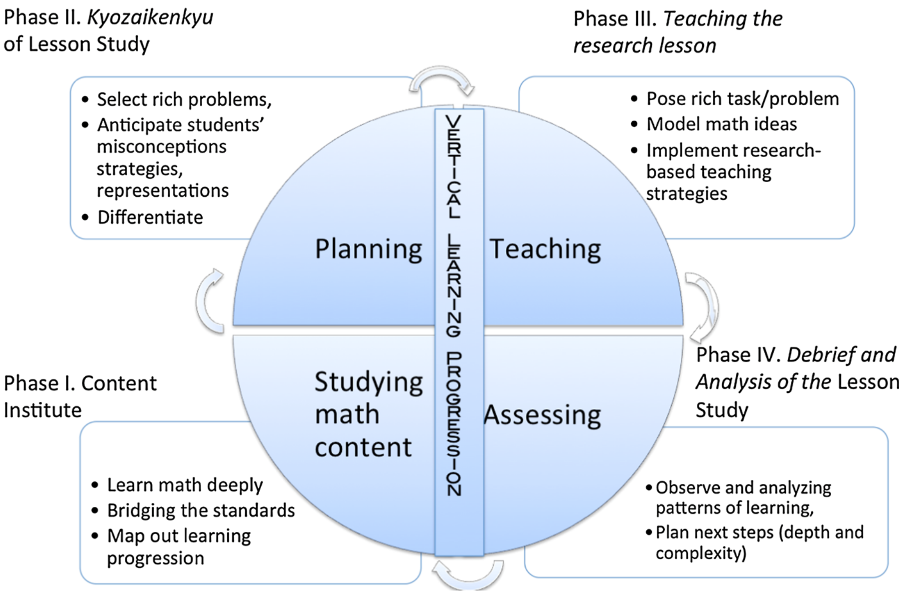


Fig. 8 Catalyst for change: understanding the mathematical learning progression

"live classroom" and make sense of students' thinking and research into how students learn and demonstrate their understanding in multiple ways. These phases can be described as the *pedagogies of enactment* that Grossman and McDonald (2008) describe as the learning of teaching that takes place as teachers work together to plan for, rehearse and enact the aspects of instructional practice.

The work that the teachers began during the Mathematics Content Institute carried into practice as teachers entered into the Lesson Study phase. In particular, the benchmark problem used across grade levels not only developed the discussion among teachers about the vertical progression of algebraic thinking but also the need for multiple representations and how such representations evolves across grade levels. At the final research lesson presentation, the team presented their collective learning. The third-grade teacher explained how her students observed patterns in the tables the students had created (such as in Fig. 4), the sixth- and eighth-grade teachers shared their students' work with the table, equations and graphs (see Figs. 5, 6, 7) and started discussing the notion of rate of change which led to the discussion of "slope." Following this, they talked about "steepness" which helped the third-grade teachers connect the tabular approach to the graphical approach. The eighth-grade teacher also shared how she could potentially use this activity in her classroom for a stock market game the students were involved in and connect the activity to the topic on linear equations and system of equations that she will teach later in the year.

We discovered that the vertical team afforded many professional development opportunities for teachers to think beyond their grade level. One of the constraints that we, as professional development designers, had to work with was the lack of time teachers had for this type of vertical teaming. It was with the support of a grant that we were able to "buy" teachers some time to observe other teachers teaching the research lesson in a different grade level. Teachers need more release time or professional development opportunities to observe other teachers teach with a focus on understanding the learning progression.

Another constraint that we had to work through was the rigidity of the lock-step pacing guides that teachers had to adhere to in their teaching schedule. The richness of this research lesson was that all the teachers took the same algebraic problem and geared it up or down to their grade level to experience the learning progression. We recognize that the infrastructure may not currently be in place and that more support is needed for teachers to gain access to these types of rich experiences; however, it was evident in teachers' reflection and debrief that they valued watching the learning progression "live" by observing students engaged in the task.

Implications for designing professional development to supporting teachers' understanding of the learning progression

The vertical teaming during Lesson Study and a focus on the Learning progression allowed teachers to revisit the conceptual principles underlying a concept across grade levels and anticipate strategies, representations, multiple models and misconceptions that students would encounter. They identified meaningful distinctions in the concept of setting up an equation and the multiple levels this concept can be accessed across the learning progression. Teachers saw that there was a coherent structure, recognizing that there was a pattern in the development of mathematical ideas as a concepts became more complex while knowing what underlying concepts are in between to bridge the gaps between the standards.

Some important implications for promoting the learning progression, specifically algebraic connections among elementary and middle school teachers, are summarized below. First, teachers need to re-experience algebra as learners. Teachers learned to grapple and solve algebraic problems using mathematics tools such as graphs, tables, formulas, pictures and technology and to evaluate multiple strategies with colleagues. Teachers reflected on the importance of designing rich problems that elicit algebraic reasoning and unpacked the metacognitive processes and the mathematical concepts important within these problems. As teachers used a variety of mathematics representational tools (graphs, tables, equations, diagrams, technology) and shared multiple strategies to solve problems, they developed representational fluency and recognized that various tools and representations were better and more efficient for different classes of problems. It was important for teachers to grapple with the problems and experience disequilibrium. This opportunity was a "relearning" process for many who were making sense of the algebra that they had learned procedurally; void of context or real-life applications. This experience prompted teachers to rethink how they needed to pose rich problems with connection to real-life (for example, one teacher had connected this problem to the concept of "demand" and "break-even" in economics) and other mathematics topics to prepare tools/representations for thinking and to engage students to think critically through a problem and not simplify or strip down the mathematics.

Another implication is the importance of collaboration and vertical articulation, which came to the forefront as teachers discussed problems with their colleagues and talked about the grade level mathematics expectations. This critical activity also elicited teachers' responses to recognizing the importance of breaking down the essential mathematical learning, identifying the common student misconceptions and scaffolding and differentiating for diverse learners. Teachers recognized the need to identify common student misconceptions, which would help scaffold and differentiate for diverse learners. They recognized the value of learning with and from others and comparing various strategies to extend students' thinking. In the second phase of the project, teachers shared strategies and learned from each other through Lesson Study teams.

The third and most important implication to making a reform practice "stick" is to have teachers experience success in using the approach in a real classroom through the Lesson Study to develop productive dispositions toward a reform practice. One of the primary pedagogical strategies that the professional development promoted was the use of multiple representations to make algebraic connections and enhance mathematics discourse (Stein et al. 2008) in the classroom. Teachers were intentional in building collective mathematical knowledge in the classroom by incorporating student-generated representations as tools for discussing reasoning and proof. Teachers recognized the value of engaging students in mathematics discourse by highlighting strategies and through posing higher-level questions. One of the most powerful professional learning activities was the joint reflection and debrief sessions where teachers shared evidence of success through examples from student work and analysis of student thinking. Teachers commented on the value of having the opportunity to observe one of their colleagues implementing the reform-oriented practices in a real classroom, some at their grade level and others below and above the grade level they currently taught. This unique opportunity helped teachers see, hear and notice the algebraic connections in terms of both the students' learning trajectory and the instructional progression. The observation and the debriefing engaged teachers in conversations and in collaborative analysis and interpretation of the mathematics teaching and learning that took place with the jointly planned research lesson. Analyzing students learning progression and discussing previous and future learning goals allowed teachers to make deeper vertical connections. Discussion relating to the sequence of lessons allowed them to generate related problems that would help students make horizontal connections, helping them transfer what they previously learned to similar and related problems.

This study revealed that working in vertical teams during Lesson Study allowed teachers to understand what "early algebra" sounded like, felt like, and looked like in an elementary and middle school classroom with real students. It created a vertical map to provide "a description of skills understanding and knowledge in the sequence in which they typically develop: a picture of what it means to 'improve' in an area of learning" (Masters and Forster 1996, p. 1). Participants' productive disposition toward teaching algebraic reasoning through problem solving and through multiple representations echoes the productive disposition we want to encourage in our students as described by the National Research Council (2001) as the "the inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy"(p. 116). We conclude that teachers need more professional learning experiences to develop and evaluate effective educational approaches to improve the learning and teaching of algebra, where they can observe how elementary and middle school students' transition from arithmetic to algebraic reasoning. We found that the designed learning opportunities including the content-focused institute and Lesson Study with teachers from multi-grade levels allowed for important vertical articulation to occur naturally and deeply about the algebraic connections in the curriculum they teach.

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