35. From *Measurement of a Circle:* Proposition 3*

(Approximation of π Using in Essence Upper and Lower Limits)

ARCHIMEDES

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. . . .]

I. Let AB be the diameter of any circle, O its centre, AC the tangent at A; and let the angle AOC be one-third of a right angle.

Then $\quad OA : AC\ [= \sqrt{3} : 1] > 265 : 153$ .... (1),

and $\quad OC : CA\ [= 2 : 1] = 306 : 153$ .... (2).

First, draw OD bisecting the angle AOC and meeting AC in D.

Now $\quad CO : OA = CD : DA,$ [Eucl. VI. 3]

so that $\quad [CO + OA : OA = CA : DA,$ or]

$\quad CO + OA : CA = OA : AD.$

Therefore [by (1) and (2)]

$\quad OA : AD > 571 : 153$ .... (3).

Hence $\quad OD^2 : AD^2\ [= (OA^2 + AD^2) : AD^2$

$\quad > (571^2 + 153^2) : 153^2]$

$\quad > 349450 : 23409,$

so that $\quad OD : DA > 591\frac{1}{8} : 153$ .... (4).

Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then $\quad DO : OA = DE : EA,$

so that $\quad DO + OA : DA = OA : AE.$]

Therefore $\quad OA : AE\ [> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$\quad > 1162\frac{1}{8} : 153$ .... (5).

[It follows that

$\quad OE^2 : EA^2 > \{(1162\frac{1}{8})^2 + 153^2\} : 153^2$

$\quad > (1350534\frac{33}{64} + 23409) : 23409$

$\quad > 1373943\frac{33}{64} : 23409.$]

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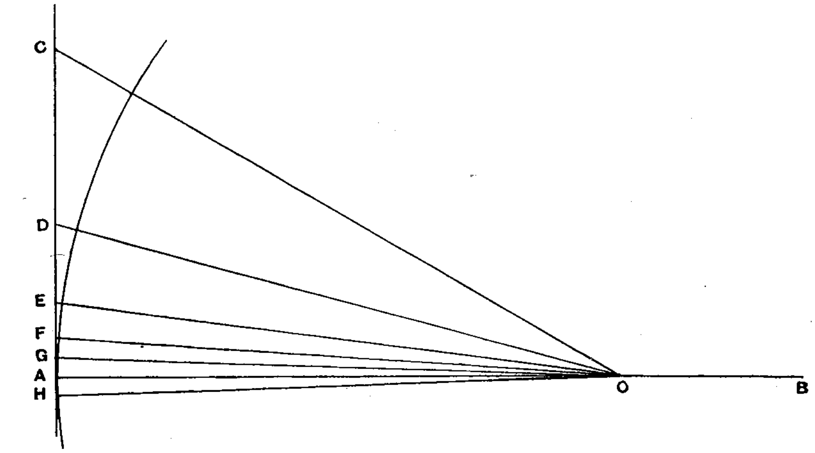


Thus $\quad OE : EA > 1172\frac{1}{8} : 153$ .... (6).

Thirdly, let OF bisect the angle AOE and meet AE in F.

We thus obtain the result [corresponding to (3) and (5) above] that

$\quad OA : AF\ [> (1162\frac{1}{8} + 1172\frac{1}{8}) : 153]$

$\quad > 2334\frac{1}{4} : 153$ .... (7).

[Therefore $\quad OF^2 : FA^2 > \{(2334\frac{1}{4})^2 + 153^2\} : 153^2$

$\quad > 5472132\frac{1}{16} : 23409.$]

Thus $\quad OF : FA > 2339\frac{1}{4} : 153$ .... (8).

Fourthly, let OG bisect the angle AOF, meeting AF in G.

We have then

$\quad OA : AG\ [> (2334\frac{1}{4} + 2339\frac{1}{4}) : 153$, by means of (7) and (8)]

$\quad > 4673\frac{1}{2} : 153.$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48} \text{ (a right angle)}.$$

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then $\quad \angle GOH = \frac{1}{24}$ (a right angle).

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $\quad OA : AG > 4673\frac{1}{2} : 153,$

while $\quad AB = 2OA, \quad GH = 2AG,$

it follows that

$\quad AB$: (perimeter of polygon of 96 sides) $[> 4673\frac{1}{2} : 153 \times 96]$

$\quad > 4673\frac{1}{2} : 14688.$

But
$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$
$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter AB.

II. Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle. Join BC.

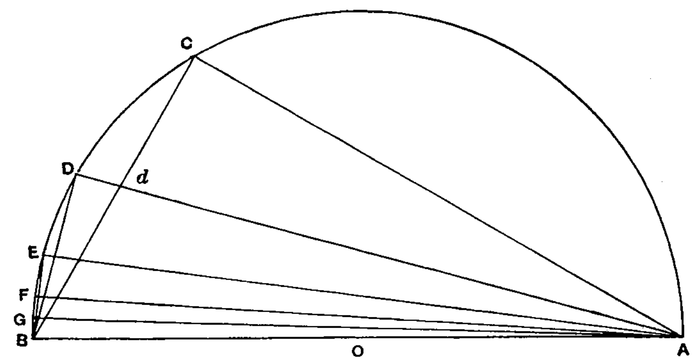
Then $AC : CB\ [= \sqrt{3} : 1] < 1351 : 780.$

First, let AD bisect the angle BAC and meet BC in d and the circle in D. Join BD.

Then $\angle BAD = \angle dAC$
$= \angle dBD,$

and the angles at D, C are both right angles.

It follows that the triangles ADB, $[ACd]$, BDd are similar.



Therefore $AD : DB = BD : Dd$
$[= AC : Cd]$
$= AB : Bd$ [Eucl. VI. 3]
$= AB + AC : Bd + Cd$
$= AB + AC : BC$

or $BA + AC : BC = AD : DB.$

[But $AC : CB < 1351 : 780$, from above,

while $BA : BC = 2 : 1$
$= 1560 : 780.]$

Therefore $AD : DB < 2911 : 780$ .... (1).

[Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$
$< 9082321 : 608400.]$

Thus $AB : BD < 3013\tfrac{3}{4} : 780$ .... (2).

Secondly, let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

Then we prove, in the same way as before, that

$AE : EB\ [= BA + AD : BD$
$< (3013\tfrac{3}{4} + 2911) : 780$, by (1) and (2)]
$< 5924\tfrac{3}{4} : 780$
$< 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13}$
$< 1823 : 240$ .... (3).

[Hence $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$
$< 3380929 : 57600.]$

Therefore $AB : BE < 1838\tfrac{9}{11} : 240$ .... (4).

Thirdly, let AF bisect the angle BAE, meeting the circle in F.

Thus $AF : FB\ [= BA + AE : BE$
$< 3661\tfrac{9}{11} : 240$, by (3) and (4)]
$< 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40}$
$< 1007 : 66$ .... (5).

[It follows that

$AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$
$< 1018405 : 4356.]$

Therefore $AB : BF < 1009\tfrac{1}{6} : 66$ .... (6).

Fourthly, let the angle BAF be bisected by AG meeting the circle in G.

Then $AG : GB\ [= BA + AF : BF]$
$< 2016\tfrac{1}{6} : 66$, by (5) and (6).

[And $AB^2 : BG^2 < \left\{(2016\tfrac{1}{6})^2 + 66^2\right\} : 66^2$
$< 4069284\tfrac{1}{36} : 4356.]$

Therefore $AB : BG < 2017\tfrac{1}{4} : 66,$

whence $BG : AB > 66 : 2017\tfrac{1}{4}$ .... (7).

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre is
$\tfrac{1}{24}$ (a right angle).]

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

(perimeter of polygon) : $AB\ [> 96 \times 66 : 2017\tfrac{1}{4}]$
$> 6336 : 2017\tfrac{1}{4}.$

And $$\frac{6336}{2017\tfrac{1}{4}} > 3\tfrac{10}{71}.$$

Much more then is the circumference to the diameter
$< 3\tfrac{1}{7}$ but $> 3\tfrac{10}{71}$.

36. From *Quadrature of the Parabola:* Introduction, Propositions 20, 23, and 24*

ARCHIMEDES

INTRODUCTION

Archimedes to Dositheus greeting.

When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square

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