

It is then possible to find two lines  $\beta, \gamma$ , of which  $\beta$  is the greater, such that  $\beta : \gamma < (\text{surface of sphere}) : C$  [Prop. 2]. Take such lines, and let  $\delta$  be a mean proportional between them.

Suppose similar regular polygons with  $4n$  sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio  $\beta : \delta$  [Prop. 3].

Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolutions as before.

Then (surface of outer solid) : (surface of inner solid) = (side of outer)<sup>2</sup> : (side of inner)<sup>2</sup> [Prop. 32]  $< \beta^2 : \delta^2$ , or  $\beta : \gamma < (\text{surface of sphere}) : C$ , a fortiori.

But this is impossible, since the surface of the circumscribed solid is greater than that of the sphere [Prop. 28], while the surface of the inscribed solid is less than  $C$  [Prop. 25].

Therefore  $C$  is not less than the surface of the sphere.

**II.** Suppose  $C$  greater than the surface of the sphere.

Take lines  $\beta, \gamma$ , of which  $\beta$  is the greater, such that  $\beta : \gamma < C : (\text{surface of sphere})$ .

Circumscribe and inscribe to the great circle similar regular polygons, as before, such that their sides are in a ratio

less than that of  $\beta$  to  $\delta$ , and suppose solids of revolution generated in the usual manner. Then, in this case, (surface of circumscribed solid) : (surface of inscribed solid)  $< C : (\text{surface of sphere})$ .

But this is impossible, because the surface of the circumscribed solid is greater than  $C$  [Prop. 30], while the surface of the inscribed solid is less than that of the sphere [Prop. 23].

Thus  $C$  is not greater than the surface of the sphere.

Therefore, since it is neither greater nor less,  $C$  is equal to the surface of the sphere.

### PROPOSITION 34

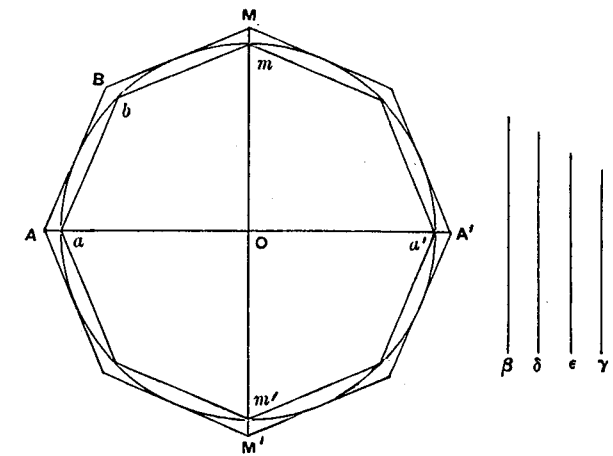
*Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.*

Let the sphere be that of which  $ama'm'$  is a great circle.

If now the sphere is not equal to four times the cone described, it is either greater or less.

**I.** If possible, let the sphere be greater than four times the cone.

Suppose  $V$  to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere.



Then, by hypothesis, the sphere is greater than  $V$ ; and two lines  $\beta, \gamma$  can be found (of which  $\beta$  is the greater) such that  $\beta : \gamma < (\text{volume of sphere}) : V$ .

Between  $\beta$  and  $\gamma$  place two arithmetic means  $\delta, \epsilon$ .

As before, let similar regular polygons with sides  $4n$  in number be circumscribed about and inscribed in the great circle, such that their sides are in a ratio less than  $\beta : \delta$ .

Imagine the diameter  $aa'$  of the circle to be in the same straight line with a diameter of both polygons, and imagine the latter to revolve with the circle about  $aa'$ , describing the surfaces of two solids of revolution. The volumes of these solids are therefore in the triplicate ratio of their sides [Prop. 32].

Thus (vol. of outer solid) : (vol. of inscribed solid)  $< \beta^3 : \delta^3$ , by hypothesis,  $< \beta : \gamma$ , a fortiori (since  $\beta : \gamma > \beta^3 : \delta^3$ ),  $< (\text{volume of sphere}) : V$ , a fortiori.

But this is impossible, since the volume of the circumscribed solid is

greater than that of the sphere [Prop. 28], while the volume of the inscribed solid is less than  $V$  [Prop. 27].

Hence the sphere is not greater than  $V$ , or four times the cone described in the enunciation.

**II.** If possible, let the sphere be less than  $V$ . In this case we take  $\beta, \gamma$  ( $\beta$  being the greater) such that  $\beta : \gamma < V : (\text{volume of sphere})$ .

The rest of the construction and proof proceeding as before, we have finally (volume of outer solid) : (volume of inscribed solid)  $< V : (\text{volume of sphere})$ .

But this is impossible, because the volume of the outer solid is greater than  $V$  [Prop. 31, Cor.], and the volume of the inscribed solid is less than the volume of the sphere.

Hence the sphere is not less than  $V$ .

Since then the sphere is neither less nor greater than  $V$ , it is equal to  $V$ , or to four times the cone described in the enunciation.