ON EQUIPOLLENCES OF NUMBERS

. . . And in order better to understand what has been said above about this art and style of abbreviating and equating its terms [*parties*] and of bringing them back to two simple terms as well as one can do it, we shall give here some examples of which the first is as follows—I shall abbreviate:

$$℞^2\ \underline{4^{2}.\tilde{p}.4^{1}}\ \tilde{p}.2^{1}.\tilde{p}.1 \quad \text{equal to } .100.^{3}$$

First I take away $.2^{1}.\tilde{p}.1$ from both terms and there remains to me $℞^2\ \underline{4^{2}.\tilde{p}.4^{1}}$ in one term and $99.\tilde{m}.2^{1}.$ in the other. And now that one of the terms is a second root it is convenient to multiply it by itself and we obtain $.4^{2}.1.\tilde{p}.4^{1}$ in this term. And similarly we must multiply $.99.\tilde{m}.2^{1}.$ by itself and we obtain $9801.\tilde{m}.396^{1}.\tilde{p}.4^{2}.$ in the other term. Now we still must abbreviate these terms by taking away $.4^{2}.$ from the one and the other term. And then add $.396^{1}.$ to each of them. In this way we shall have $.400^{1}.$ in one term and $.9801.$ in the other term.[4]

After more of this there follows a theory of quadratic equations, in which negative roots are rejected. Chuquet has negative but no fractional exponents. Those we meet, even before Chuquet, in the *Algorismus proportionum* by Nicole Oresme (*c.* 1323–1382; see Selection III.1). Here we find a notation $\boxed{\begin{array}{c}1\\ \hline 2\end{array}}$ for $\frac{1}{2}$, $\boxed{\begin{array}{c}p\ \ 1.\\ \hline 1\ \ .2.\end{array}}$ for $1\frac{1}{2}$, $\boxed{\begin{array}{c}p\ 1\\ \hline 1\ 3\end{array}}$ for $1\frac{1}{3}$, $\boxed{\begin{array}{c}p\ 2\\ \hline 2\ 4\end{array}}$ for $2\frac{2}{4}$, and $\boxed{\begin{array}{c}p\ 1.\\ \hline 1\ 2.\end{array}}$ 4 for $4^{1\frac{1}{2}}$. The *p* stands for *proportio*. The dots (.) are sometimes present, sometimes absent in the manuscript text reproduced by F. Cajori, *History of mathematical notations* (Open Court, Chicago, 1928), I, 92. A variant $\boxed{1^{p}\frac{1}{2}}$ 4 is found in Cantor, *Geschichte*, II, 121. On the *Algorismus* see *De proportionibus proportionum and Ad pauca respicientes*, ed. E. Grant (University of Wisconsin Press, Madison, 1966), 65–68.

3 CARDAN. ON CUBIC EQUATIONS

The discovery of the numerical solutions of equations of the third degree at the University of Bologna in the early years of the sixteenth century was an important step in the development of algebra. It attracted wide attention, and was discussed in many public disputations. The textbook that laid the whole method open to public inspection was the *Ars magna* (Nuremberg, 1545) by the physician, humanist, mathematician, and scientist-in-general Gerolamo Cardano, or Hieronymus Cardanus, or, in English, Jerome Cardan (1501–1576). Here he stated that Scipio del Ferro at Bologna had discovered the method of solving equations of the type $x^3 + px = q$. Nicolo Tartaglia (*c.* 1499–1557) had also discovered this

[3] $\sqrt{4x^2 + 4x} + 2x + 1 = 100.$

[4] $\sqrt{4x^2 + 4x} = 99 - 2x$; $4x^2 + 4x = 9801 - 396x + 4x^2$; $4x = 9801 - 396x$; $400x = 9801$

and then found a method of solving equations of the type $x^3 = px + q$, $x^3 + q = px$. Cardan obtained the solutions from Tartaglia (breaking a pledge of secrecy) and the method of solving cubic equations numerically has ever since been called after Cardan. The *Ars magna* was for many decades the best-known book on algebra, studied by all who were interested, and it lost this position only when Descartes introduced his new methods.

We quote here an English translation of a part of Chapter XI (pp. 29^{r}–30^{r}), dealing with the equation $x^3 + px = q$, or in particular $x^3 + 6x = 20$. It is based, as is also the text of Selection II.4, on the translation published in Smith, *Source book*, 204–212. Cardan's notation is quite different from ours, and he expresses the equation by saying: "A cube and unknowns are equal to a number" (Cubus et res aequales numero). For "unknown," our x, he has, like most of his contemporaries, the Latin term *res*, Italian *cosa*, literally, "thing." A cube is conceived as a solid body. By "number" is meant a numerical coefficient, in this case 20.

The book contains solutions for quadratics and for many types of cubes and biquadratics. The coefficients are always positive and specific numbers. Cardan also teaches some properties of equations and their roots. For instance (in Chapter XVII) we read that the equation $x^3 + 10x = 6x^2 + 4$ has three roots, namely 2, $2 + \sqrt{2}$, $2 - \sqrt{2}$, and Cardan sees that their sum adds up to the coefficient of x^2. Cardan is puzzled when imaginaries appear, and keeps them out of the *Ars magna* except in one case (see below), where he meets them in the solution of a quadratic equation. The *casus irreducibilis*, where a real root appears as a sum of the cube roots of two imaginaries (as in $x^3 = 15x + 4$, where $x = 4$, but the Cardan formula gives $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$) is discussed in the works of Bombelli (1572) and Viète (1591).

On Cardan see O. Ore, *Cardano, the gambling scholar* (Princeton University Press, Princeton, New Jersey, 1953). On the *Ars magna* see J. F. Scott, *A history of mathematics* (Taylor and Francis, London, 1958), 87–92. On Italian mathematicians of the Renaissance, see further E. Bortolotti, *Studi e ricerche sulla storia della matematica in Italia nei secoli XVI e XVII* (Zanichelli, Bologna, 1928).

CONCERNING A CUBE AND UNKNOWNS EQUAL TO A NUMBER

Chapter XI

Scipio del Ferro of Bologna about thirty years ago invented [the method set forth in] this chapter, [and] communicated it to Antonio Maria Florido of Venice, who when he once engaged in a contest with Nicolo Tartaglia of Brescia announced that Nicolo also invented it: and he [Nicolo] communicated it to us when we asked for it, but suppressed the demonstration.[1] With this aid we sought the demonstration, and found it, though with great difficulty, in the manner which we set out in the following.

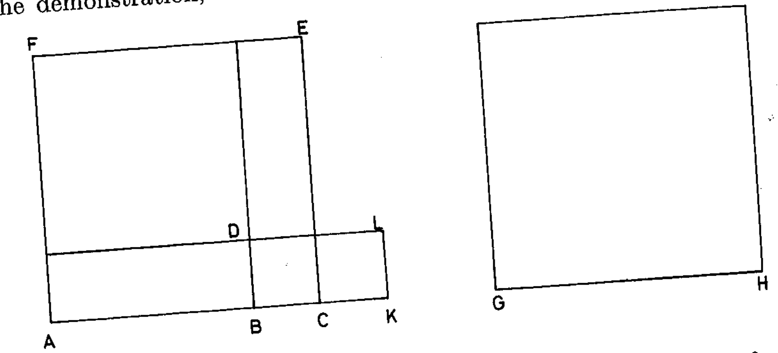
Demonstration. For example, let the cube of GH and six times the side GH be

[1] Tartaglia and Cardan met in Milan during 1539, after which Tartaglia gave Cardan his method in obscure verses, which he later clarified. Cardan soon mastered the method and knew how to apply it independently. The verses begin as follows:

Quando che'l cubo con le cose appresso	When x^3 together with px
Se agguaglia a qualche numero discreto	Are equal to a q
Trovan dui altri, differenti in esso . . .	Then take u and v, $u \neq v$ . . .

equal to 20.[2] I take [Fig. 1] two cubes AE and CL whose difference shall be 20, so that the product of the side AC by the side CK shall be 2, i.e., a third of the number of unknowns, and I lay off CB equal to CK; then I say that if it is done thus, the remaining line AB is equal to GH and therefore to the value of the unknown (for it was supposed of GH that it was so). Therefore I complete, after the manner of the first theorem of the 6th chapter of this book,[3] the solids DA, DC, DE, DF, so that we understand by DC the cube of BC, by DF the cube of AB, by DA three times CB times the square of AB, by DE three times AB times the square of BC. Since therefore from AC times CK the result is 2, from 3 times AC times CK will result 6, the number of unknowns, and therefore from AB times 3 AC times CK there results 6 unknowns AB, or 6 times AB, so that 3 times the product of AB, BC, and AC is 6 times AB. But the difference of the cube AC from the cube CK, and likewise from the cube BC, equal to it by hypothesis, is 20; and from the first theorem of the 6th chapter, this is the sum of the solids DA, DE, and DF, so that these three solids make 20. But taking BC minus, the cube of AB is equal to the cube of AC and 3 times AC into the square of CB and minus the cube of BC and minus 3 times BC into the square of AC. By the demonstration, the difference between 3 times CB times the

Fig. 1



[2] We can follow the reasoning more easily if we take the equation as $x^3 + px = q$, $p = 6$, $q = 20$, and $GH = x$, $AC = u$, $CK = v$, so that the cube $AE = u^3$, $DC = v^3$. Then u and v are selected so that $u^3 - v^3 = q = 20$, $uv = p/3 = 2$. Then we must prove that $AB = u - v = GH = x$. For this purpose we use for $AE = u^3$ the expression for the third power of the binomial ("the first theorem of the 6th chapter," this theorem being stated as a property of solids):

$$u^3 = [(u - v) + v]^3 = (u - v)^3 + 3v(u - v)^2 + 3v^2(u - v) + v^3$$

so that

$$u^3 - v^3 = (u - v)^3 + 3v(u - v)^2 + 3(u - v)v^2 = (u - v)^3 + 3uv(u - v),$$

or

$$u^3 - v^3 = q = (u - v)^3 + p(u - v).$$

Hence, since $q = x^3 + px$ (here Cardan quotes theorems in Euclid's *Elements*, book XI, dealing with the equality of parallelepipeds; the numbering of the propositions differs in ancient editions), we see that $x = u - v$, or $AB = GH$.

[3] Cardan writes in Chapter VI that after Tartaglia had handed over to him his rule he "thought that this would be the royal road to pursue in all cases." And so he established three theorems; in our notation they are:

(a) If $a = u + v$, then $a^3 = u^3 + v^3 + 3(u^2v + uv^2)$;
(b) $u^3 + 3uv^2 > v^3 + 3u^2v$, the difference being $(u - v)^3 (u > v)$;
(c) By Euclid's theory of proportions,

$$\frac{u^3 + v^3}{3uv^2 + 3vu^2} = \frac{u^3 - u^2v + uv^2}{3vu^2}$$

square of AC, and 3 times AC times the square of BC, is [3 times] the product of AB, BC, and AC. Therefore since this, as has been shown, is equal to 6 times AB, adding 6 times AB to that which results from AC into 3 times the square of BC there results 3 times BC times the square of AC, since BC is minus. Now it has been shown that the product of CB[4] into 3 times the square of AC is minus; and the remainder which is equal to that is plus, hence 3 times CB into the square of AC and 3 times AC into the square of CB and 6 times AB make nothing. Accordingly, by common sense, the difference between the cubes AC and BC is as much as the totality of the cube of AC, and 3 times AC into the square of CB, and 3 times CB into the square of AC (minus), and the cube of BC (minus), and 6 times AB. This therefore is 20, since the difference of the cubes AC and CB was 20. Moreover, by the second theorem of the 6th chapter, putting BC minus, the cube of AB will be equal to the cube of AC and 3 times AC into the square of BC minus the cube of BC and minus 3 times BC into the square of AC. Therefore the cube of AB, with 6 times AB, by common sense, since it is equal to the cube of AC and 3 times AC into the square of CB, and minus 3 times CB into the square of AC, and minus the cube of CB and 6 times AB, which is now equal to 20, as has been shown, will also be equal to 20. Since therefore the cube of AB and 6 times AB will equal 20, and the cube of GH, together with 6 times GH, will equal 20, by common sense and from what has been said in the 35th and 31st of the 11th Book of the *Elements*, GH will be equal to AB, therefore GH is the difference of AC and CB. But AC and CB, or AC and CK, are numbers or lines containing an area equal to a third part of the number of unknowns whose cubes differ by the number in the equation, wherefore we have the

RULE[5]

Cube the third part of the number of unknowns, to which you add the square of half the number of the equation, and take the root of the whole, that is, the

[4] Here begins the text of p. 30r of the *Ars magna*, reproduced in Fig. 2.

[5] This rule is known as Cardan's rule for the case $x^3 + px = q$. In our notation:
Since $u^3 - v^3 = q = 20$, $uv = p/3$, we can find $x = u - v$ by solving a quadratic equation. Since $v = p/3u$, $u^3 - (p/3u)^3 = q$, $u^6 - qu^3 - (p/3)^3 = u^6 - 20u^3 - p = 0$, we find

$$u^3 = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)} = 10 \pm \sqrt{100 + 8} = 10 \pm \sqrt{108}$$

Similarly:

$$v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = -10 \pm \sqrt{108}.$$

Cardan now states in the "Rule": for u^3 take $10 + \sqrt{108}$ (this is the *binomial*), for v^3 take $-10 + \sqrt{108}$ (this is the *apotome*; both expressions are from Euclid's *Elements*, Book X), so that

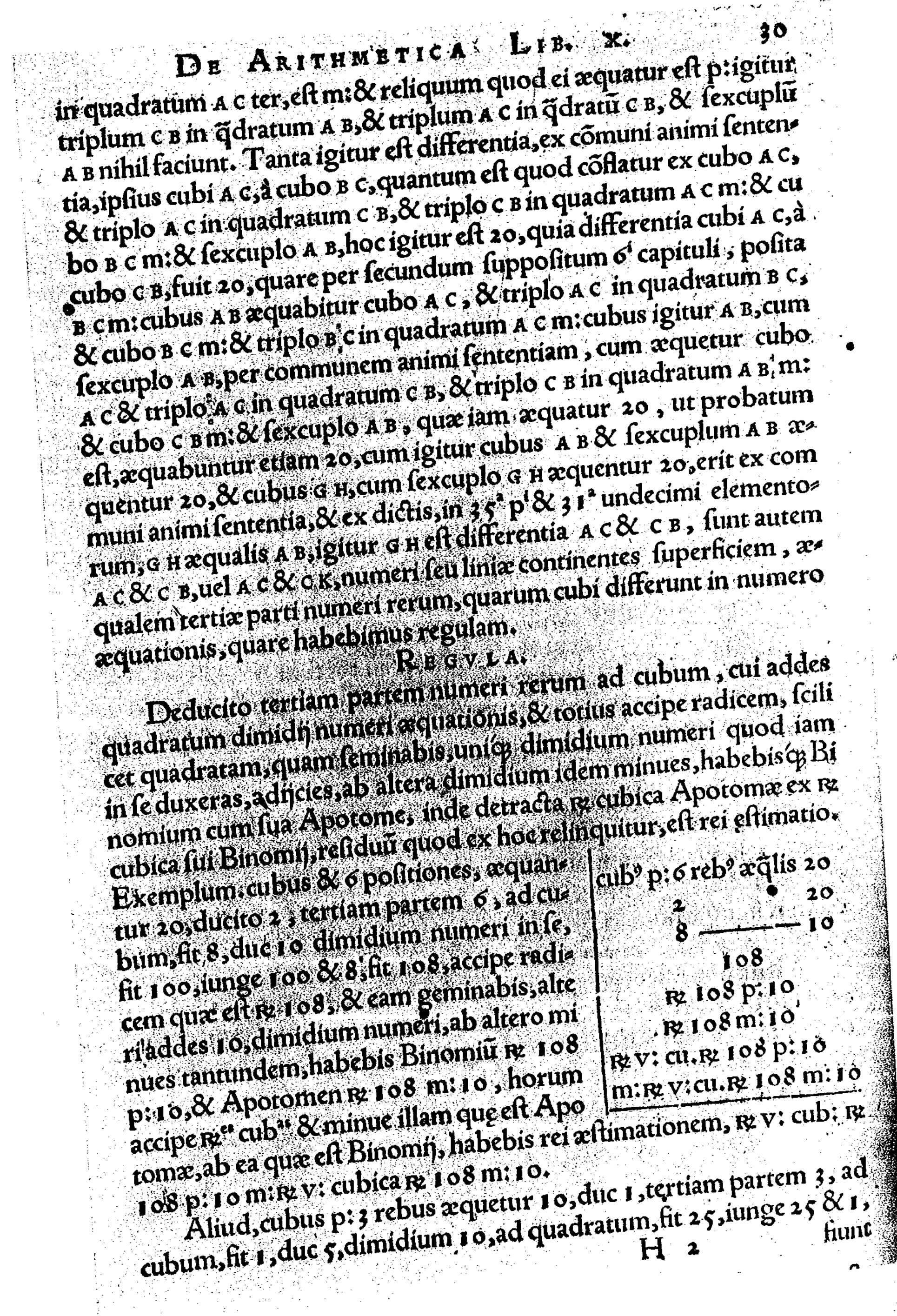
$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$= \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}.$$

In Cardan's notation (see Fig. 2, fourth line from the bottom):

℞ V: cub: ℞ 108 *p*: 10 *m*: ℞ V: cubica ℞ 108 *m*: 10;

here *p* stands for "piu," plus, *m* for "meno," minus, and ℞ for "radix." Cardan does not use the signs +, −, although they were already in use at the time.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in quadratum A C ter, est m: & reliquum quod ei æquatur est p: igitur
triplum C B in q̃dratum A B, & triplum A C in q̃dratũ C B, & sexcuplũ
A B nihil faciunt. Tanta igitur est differentia, ex cõmuni animi senten-
tia, ipsius cubi A C, à cubo B C, quantum est quod cõflatur ex cubo A C,
& triplo A C in quadratum C B, & triplo C B in quadratum A C m: & cu
bo B C m: & sexcuplo A B, hoc igitur est 20, quia differentia cubi A C, à
cubo C B, fuit 20, quare per secundum suppositum 6[i] capituli, posita
B C m: cubus A B æquabitur cubo A C, & triplo A C in quadratum B C,
& cubo B C m: & triplo B C in quadratum A C m: cubus igitur A B, cum
sexcuplo A B, per communem animi sententiam, cum æquetur cubo
A C & triplo A C in quadratum C B, & triplo C B in quadratum A B, m:
& cubo C B m: & sexcuplo A B, quæ iam æquatur 20, ut probatum
est, æquabuntur etiam 20, cum igitur cubus A B & sexcuplum A B æ-
quentur 20, & cubus G H, cum sexcuplo G H æquentur 20, erit ex com
muni animi sententia, & ex dictis, in 35[a] p[i] & 31[a] undecimi elemento-
rum, G H æqualis A B, igitur G H est differentia A C & C B, sunt autem
A C & C B, uel A C & C K, numeri seu liniæ continentes superficiem, æ-
qualem tertiæ parti numeri rerum, quarum cubi differunt in numero
æquationis, quare habebimus regulam.

Regula.

Deducito tertiam partem numeri rerum ad cubum, cui addes
quadratum dimidij numeri æquationis, & totius accipe radicem, scili
cet quadratam, quam geminabis, uniq; dimidium numeri quod iam
in se duxeras, adijcies, ab altera dimidium idem minues, habebisq; Bi
nomium cum sua Apotome, inde detracta ℞ cubica Apotomæ ex ℞
cubica sui Binomij, residuũ quod ex hoc relinquitur, est rei estimatio.
Exemplum. cubus & 6 positiones, æquan-
tur 20, ducito 2, tertiam partem 6, ad cu-
bum, fit 8, duc 10 dimidium numeri in se,
fit 100, iunge 100 & 8, fit 108, accipe radi-
cem quæ est ℞ 108, & eam geminabis, alte
ri addes 10, dimidium numeri, ab altero mi
nues tantundem, habebis Binomiũ ℞ 108
p: 10, & Apotomen ℞ 108 m: 10, horum
accipe ℞[ae] cub[as] & minue illam quę est Apo
tomæ, ab ea quæ est Binomij, habebis rei æstimationem, ℞ v: cub: ℞
108 p: 10 m: ℞ v: cubica ℞ 108 m: 10.

cub[9] p: 6 reb[9] æq̃lis 20
2 20
8 —— 10
108
℞ 108 p: 10
℞ 108 m: 10
℞ v: cu. ℞ 108 p: 10
m: ℞ v: cu. ℞ 108 m: 10

Aliud, cubus p: 3 rebus æquetur 10, duc 1, tertiam partem 3, ad
cubum, fit 1, duc 5, dimidium 10, ad quadratum, fit 25, iunge 25 & 1,
fiunt

H 2

Fig. 2

square root, which you will use, in the one case adding the half of the number which you just multiplied by itself, in the other case subtracting the same half, and you will have a binomial and apotome respectively; then subtract the cube root of the apotome from the cube root of the binomial, and the remainder from this is the value of the unknown. In the example, the cube and 6 unknowns equals 20; raise 2, the 3rd part of 6, to the cube, that makes 8; multiply 10, half the number, by itself, that makes 100; add 100 and 8, that makes 108; take the root, which is $\sqrt{108}$, and use this, in the first place adding 10, half the number, and in the second place subtracting the same amount, and you will have the binomial $\sqrt{108} + 10$, and the apotome $\sqrt{108} - 10$; take the cube root of these and subtract that of the apotome from that of the binomial, and you will have the value of the unknown $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.

Cardan continues to discuss one case after another. Here are, in our notation, the headings of the different chapters:

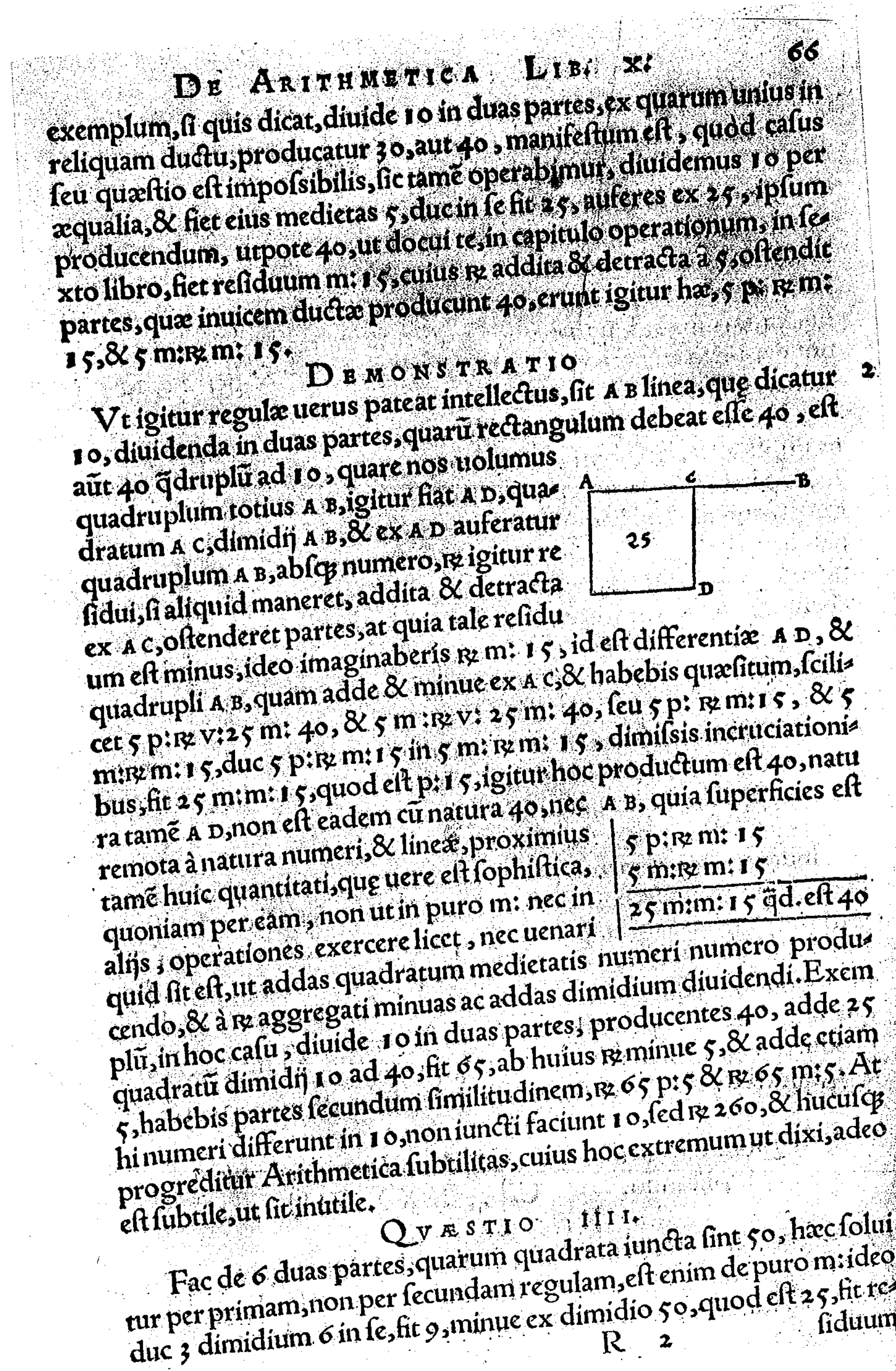
11. $x^3 + ax = b$
12. $x^3 = ax + b$
13. $x^3 + a = bx$
14. $x^3 = ax^2 + b$
15. $x^3 + ax^2 = b$
16. $x^3 + a = bx^2$
17. $x^3 + ax^2 + bx = c$
18. $x^3 + ax = bx^2 + c$
19. $x^3 + ax^2 = bx + c$
20. $x^3 = ax^2 + bx + c$
21. $x^3 + a = bx^2 + cx$
22. $x^3 + ax + b = cx^2$
23. $x^3 + ax^2 + b = cx$
24. On the 44 derivative equations (for example, $x^6 + 6x^4 = 100$)
25. On imperfect and particular rules.

Chapter 26 and later chapters also deal with biquadratic equations.

Many examples follow. We occasionally meet negative numbers, which Cardan calls "fictitious" (*fictae*). Another element enters in the following example, taken from Chapter 37, "On the rule of postulating a negative," which involves imaginaries. We substitute modern notation.

I will give as an example:[6] If some one says to you, divide 10 into two parts, one of which multiplied into the other shall produce 30 or 40, it is evident that this case or question is impossible. Nevertheless, we shall solve it in this fashion. Let us divide 10 into equal parts and 5 will be its half. Multiplied by itself, this yields 25. From 25 subtract the product itself, that is 40, which, as I taught you in the chapter on operations in the sixth book, leaves a remainder -15. The root of this added to and then subtracted from 5 gives the parts which multiplied together will produce 40. These, therefore, are $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$.

[6] Here begins the text of p. 66[r] of the *Ars magna*, reproduced in Fig. 3.



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exemplum, ſi quis dicat, diuide 10 in duas partes, ex quarum unius in
reliquam ductu, producatur 30, aut 40, manifeſtum eſt, quod caſus
ſeu quæſtio eſt impoſſibilis, ſic tamẽ operabimur, diuidemus 10 per
æqualia, & fiet eius medietas 5, duc in ſe fit 25, auferes ex 25, ipſum
producendum, utpote 40, ut docui te, in capitulo operationum, in ſe-
xto libro, fiet reſiduum m: 15, cuius ℞ addita & detracta à 5, oſtendit
partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: ℞ m:
15, & 5 m: ℞ m: 15.

DEMONSTRATIO 2

Vt igitur regulæ uerus pateat intellectus, ſit A B linea, quę dicatur
10, diuidenda in duas partes, quarũ rectangulum debeat eſſe 40, eſt
aũt 40 q̃druplũ ad 10, quare nos uolumus
quadruplum totius A B, igitur fiat A D, qua-
dratum A C, dimidij A B, & ex A D auferatur
quadruplum A B, abſq; numero, ℞ igitur re
ſidui, ſi aliquid maneret, addita & detracta
ex A C, oſtenderet partes, at quia tale reſidu
um eſt minus, ideo imaginaberis ℞ m: 15, id eſt differentiæ A D, &
quadrupli A B, quam adde & minue ex A C, & habebis quæſitum, ſcili-
cet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, ſeu 5 p: ℞ m: 15, & 5
m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimiſſis incruciationi-
bus, fit 25 m: m: 15, quod eſt p: 15, igitur hoc productum eſt 40, natu
ra tamẽ A D, non eſt eadem cũ natura 40, nec A B, quia ſuperficies eſt
remota à natura numeri, & lineæ, proximius
tamẽ huic quantitati, quę uere eſt ſophiſtica,
quoniam per eam, non ut in puro m: nec in
alijs, operationes exercere licet, nec uenari
quid ſit eſt, ut addas quadratum medietatis numeri numero produ-
cendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exem
plũ, in hoc caſu, diuide 10 in duas partes, producentes 40, adde 25
quadratũ dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam
5, habebis partes ſecundum ſimilitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At
hi numeri differunt in 10, non iuncti faciunt 10, ſed ℞ 260, & hucuſq;
progreditur Arithmetica ſubtilitas, cuius hoc extremum ut dixi, adeo
eſt ſubtile, ut ſit inutile.

5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̃d. eſt 40

QVÆSTIO IIII.

Fac de 6 duas partes, quarum quadrata iuncta ſint 50, hæc ſolui
tur per primam, non per ſecundam regulam, eſt enim de puro m: ideo
duc 3 dimidium 6 in ſe, fit 9, minue ex dimidio 50, quod eſt 25, fit re-
ſiduum

R 2

Fig. 3

Proof. That the true significance of this rule may be made clear, let the line AB [see Fig. 3], which is called 10, be the line which is to be divided into two parts whose rectangle is to be 40. Now since 40 is the quadruple of 10, we wish four times the whole of AB. Therefore, make AD the square on AC, the half of AB. From AD subtract four times AB. If there is a remainder, its root should be added to and subtracted from AC thus showing the parts [into which AB was to be divided]. Even when such a residue is negative, you will nevertheless imagine $\sqrt{-15}$ to be the difference between AD and the quadruple of AB which you should add to and subtract from AC to find what was sought. That is $5 + \sqrt{25 - 40}$ and $5 - \sqrt{25 - 40}$, or $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, we obtain $25 - (-15)$ which is $+15$. Therefore the product is 40. However, the nature of AD is not the same as that of 40 or AB because a surface is far from a number or a line. This, however, is closest to this quantity, which is truly puzzling since operations may not be performed with it as with a pure negative number or with other numbers.[7] Nor can we find it by adding the square of half the number to the product number and take away and add from the root of the sum half of the dividend. For example, in the case of dividing 10 into two parts whose product is 40, you add 25, the square of one half of 10, to 40 making 65. From the root of this subtract 5 and then add 5 and according to similar reasoning you will have $\sqrt{65} + 5$ and $\sqrt{65} - 5$. But these numbers differ by 10, and do not make 10 jointly.[8] This subtility results from arithmetic the final point of which is, as I have said, as subtile as it is useless.

4 FERRARI. THE BIQUADRATIC EQUATION

Cardan's *Ars magna* not only presented the numerical solution of cubic equations, but—to the surprise of his contemporaries—also showed how a biquadratic equation can be solved. This was accomplished by a young friend of Cardan's, Ludovico Ferrari (1522–1565), who tried his talents on the equation $x^4 + 6x^2 + 36 = 60x$. The method has since been known as the method of Ferrari.

The text begins with a square AD, of which the side AB is supposed to be itself a square, say $AB = x^2$. Added to AB is a part $BC = p = 3$. Then by means of another addition $CG = y$ the square AH is obtained. Figure 1 shows that the area $LNM = y^2 + 2yp$, where $MD = BC = p$.

[7] The sentence is: *quae vere est sophistica, quoniam per eam, non ut in puro m: nec in aliis, operationes exercere licet, nec venari quid sit.* T. R. Witmer (in a translation of the *Ars magna* to be published by the M.I.T. Press) translates this: "This truly is sophisticated, since through it one can (as one cannot in the case of a pure negative) perform operations and pursue a will-o'-the wisp."

[8] Since $x_1 + x_2 = 10$, $x_1x_2 = 40$, the equation to be solved is $x^2 - 10x + 40 = 0$, hence $x = 5 \pm \sqrt{25 - 40}$. Cardan, puzzled by this "sophisticated subtility," asks whether perhaps $x = \pm 5 + \sqrt{25 + 40}$ will do, but then $x_1 - x_2 = 10$ and not $x_1 + x_2$.

What is shown then is that, if an equation of the form $x^4 + px^2 + qx + r = 0$ is given, it can be written

$$\text{(A)} \qquad \begin{aligned}(x^2 + p + y)^2 &= p^2 + px^2 - qx - r + 2y(x^2 + p) + y^2 \\ &= x^2(p + 2y) - qx + (p^2 - r + 2py + y^2),\end{aligned}$$

so that the problem is reduced to the finding of a value of y that makes the right-hand member a square in x. This leads to an equation of the third degree in y:

$$4(p + 2y)(p^2 - r + 2py + y^2) - q^2 = 0.$$

Solution of this equation in y leads to the solution of the original biquadratic equation.

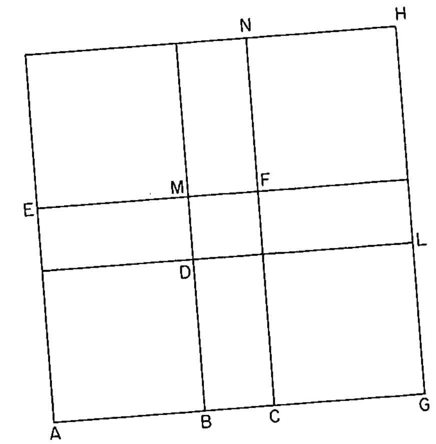


Fig. 1

If therefore AD [Fig. 1] is made 1 fourth power[1] and CD and DE are made 3 squares, and DF is made 9, BA will necessarily be a square and BC will necessarily be 3. Since we wish to add some squares to DC and DE, let these [additions] be [the rectangles] CL and KM. Then in order to complete the square it will be necessary to add the area LNM. This has been shown to consist of the square on GC, which is half the number of [added] squares, since CL is the area [made] from [the product of] GC times AB, where AB is a square, AD having been assumed to be a fourth power. But FL and MN are each equal to GC times CB, by Euclid I, 42,[2] and hence the area LMN, which is the number to be added, is a sum composed of the product of GC into twice CB, that is, into the number of squares which was 6, and GC into itself, which is the number of squares to be added. This is our proof [of the possibility of a solution].

This having been completed, you will always reduce the part containing the fourth power to a root, viz, by adding enough to each side so that the fourth power with the square and number may have a root. This is easy when you take half the number of the squares as the root of the number; and you will at the

[1] Cardan writes "square-square," quadratum quadratum (q^dq^d), hence x^4.
[2] In Heath's edition of the *Elements* (see Selection II.1) it is I, 43.

same time make the extreme terms on both sides plus, for otherwise the trinomial or binomial changed to a trinomial will necessarily fail to have a root. Having done this, you will add enough squares and a number to the one side, by the third rule,[3] so that the same being added to the other side (in which the unknowns were) will make a trinomial having a square root by assumption; and you will have a number of squares and a number to be added to each side, after which you will extract the square root of each side, which will be, on the one side, 1 square plus a number (or minus a number) and, on the other side, 1 unknown or more, plus a number (or minus a number; or a number minus unknowns), wherefore by the fifth chapter of this book you will have what has been proposed.

QUESTION V[4]

Example. Divide 10 into 3 parts in continued proportion such that the first multiplied by the second gives 6 as product. This problem was proposed by Johannes Colla,[5] who said he could not solve it. I nevertheless said I could solve it, but did not know how until Ferrari found this solution. Put then 1 unknown as the middle number, then the first will be 6/1 unknown, and the third will be $\frac{1}{6}$ of a cube. Hence these together will be equal to 10. Multiplying all by 6 unknowns we shall have 60 unknowns equal to one fourth power plus 6 squares plus 36.[6] Add, according to the 5th rule, 6 squares to each side, and you will have 1 fourth power plus 12 squares plus 36, equal to 6 squares plus 60 unknowns; for if equals are added to equals, the totals are equal. But 1 fourth power plus 12 squares plus 36 has a root, which is 1 square plus 6. If 6 squares plus 60 unknowns also had a root, we should have the job done; but they do not have; hence we must add so many squares and a number to each side, that on the one side there may remain a trinomial having a root, while on the other side it should be made so. Let therefore a number of squares[7] be an unknown and since, as you see in the figure ... CL and MK are formed from twice GC into AB, and GC is an unknown, I will always take the number of squares to be added as 2 unknowns, that is, twice GC; and since the number to be added to 36 is LNM it therefore is the square of GC together with the product of twice

[3] Rule given earlier in the book.
[4] The problem is to find $y:x = x:z$, $x + y + z = 10$, $xy = 6$, which leads to

$$\frac{6}{x} + x + \frac{1}{6}x^3 = 10.$$

This is written

$$x^4 + 12x^2 + 36 = 6x^2 + 60x, \quad \text{or} \quad (x^2 + 6)^2 = 6x^2 + 60x.$$

This is changed into $(x^2 + 6 + y)^2 = 6x^2 + 60x + 2y(x^2 + 6) + y^2$. The right-hand member is a square if

$$2y^3 + 30y^2 + 72y = 900, \quad y^3 + 15y^2 + 36y = 450,$$

or $y^3 + (12 + \frac{12}{4})y^2 + 36y = \frac{1}{2}(\frac{60}{2})^2$. This is a cubic equation, already discussed by Cardan.
[5] Zuasse de Tonini da Coi, or Johannes Colla, was a mathematician who often conferred with Tartaglia and Cardan.
[6] This means $x^4 + 6x^2 + 36 = 60x$.
[7] Here begins the text of p. 74[v], reproduced in Fig. 2.

GC into CB or of GC into twice CB, which is 12, the number of the squares in the original equation. I will therefore always multiply the unknown, half the number of squares to be added, into the number of squares in the original equation and into itself and this will make 1 square plus 12 unknowns to be added on each side, and also 2 unknowns for the number of the squares. We shall therefore have again, by common sense, the quantities written below equal to each other; and each side will have a root, the first, by the third rule,[3] but the second quantity, by an assumption [this is Eq. (A) above]. Therefore the first part of the trinomial multiplied by the third makes the square of half the second part of the trinomial. Thus from half the second part multiplied by itself there results 900, a square, and from the first [multiplied] into the third there result 2 cubes plus 30 squares plus 72 unknowns. Likewise, this may be reduced, since equals divided by equals produce equals, as 2 cubes plus 30 squares plus 72 unknowns equals 900, therefore 1 cube plus 15 squares plus 36 unknowns equals 450.

It is therefore sufficient for the reduction to the rule, if we have always 1 cube plus the number of the former squares, with a fourth of it added to it plus such a multiple of the assumed quantity as the first number of the equation indicates; so that if we had 1 fourth power plus 12 squares plus 36 equals 6 squares plus 60 unknowns we should have 1 cube plus 15 squares plus 36 unknowns equal to 450, half the square of half the number of unknowns. And if we had 1 fourth power plus 16 squares plus 64 equal to 80 unknowns we should have 1 cube plus 20 squares plus 64 unknowns equal to 800. And if we had 1 fourth power plus 20 squares plus 100 equal to 80 unknowns we should have 1 cube plus 25 squares plus 100 unknowns equal to 800. This being understood, in the former example we had 1 cube plus 15 squares plus 36 unknowns equal to 450; therefore the value of the unknown, by the 17th chapter,[8] is

$$\sqrt[3]{287\tfrac{1}{2} + \sqrt{80449\tfrac{1}{4}}} + \sqrt[3]{287\tfrac{1}{2} - \sqrt{80449\tfrac{1}{4}}} - 5.$$ [9]

This then is the number of squares which is to be doubled and added to each side (since we assumed 2 unknowns to be added), and the number to be added to each side, by the demonstration, is the square of this, with the product of this by 12, the number of squares.

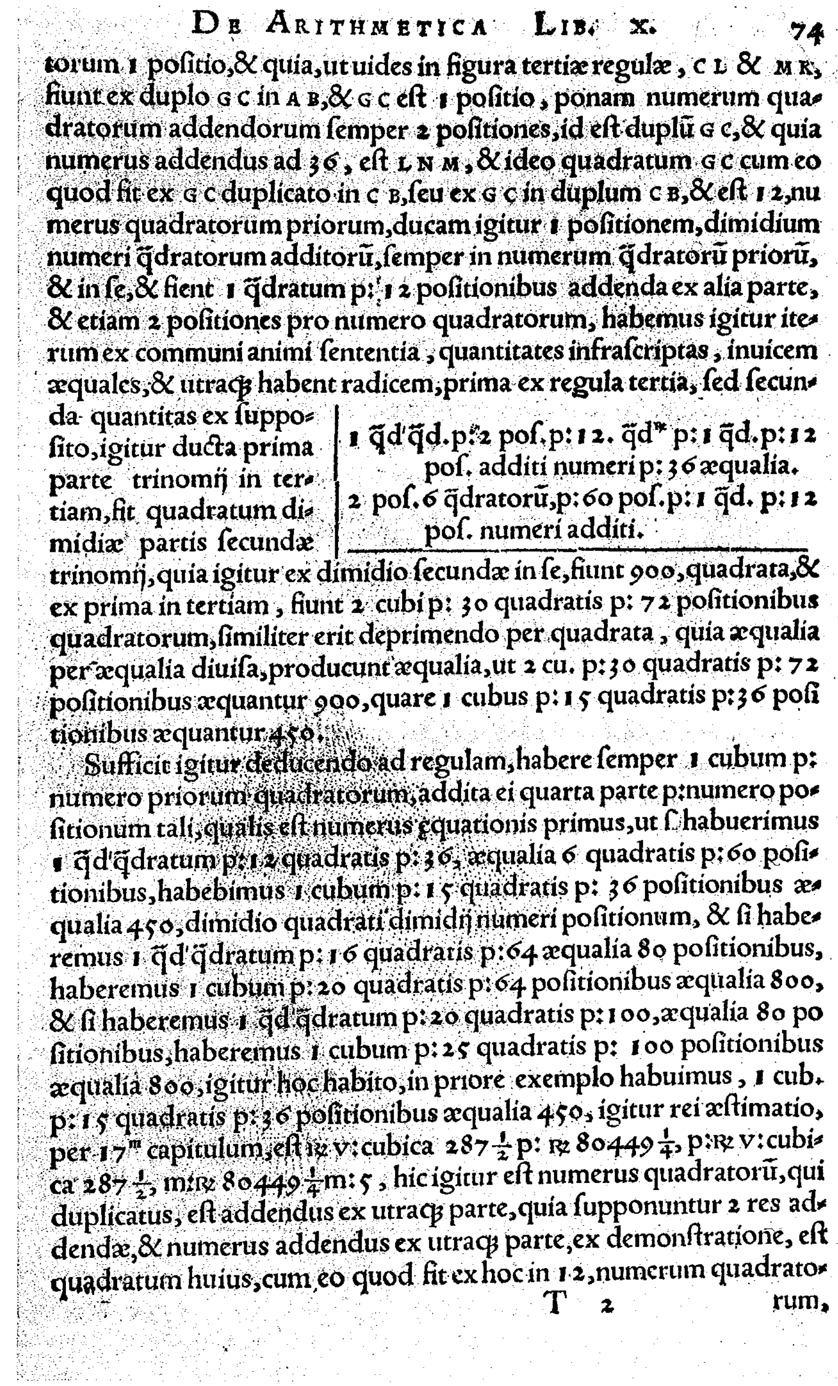
Cardan continues to analyze this method and gives several more examples, for instance $x^4 + 4x + 8 = 10x^2$, which he reduces to $y^3 + 30 = 2y^2 + 15y$. The book ends, at the conclusion of the 40th chapter, with the exclamation: "Written in five years, may it last as many thousands!"

[8] Cardan here teaches that an equation of the form $x^3 + px^2 + qx = r$ can be reduced to an equation without a term in x^2 by the substitution $y = x + p/3$.

[9] T. R. Witmer has observed that this should be

$$\sqrt[3]{190 + \sqrt{33{,}903}} + \sqrt[3]{190 - \sqrt{33{,}903}} - 5$$

(see note 7, previous selection).



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torum 1 poſitio,& quia,ut uides in figura tertiæ regulæ, C L & M K, fiunt ex duplo G C in A B,& G C eſt 1 poſitio, ponam numerum quadratorum addendorum ſemper 2 poſitiones,id eſt duplū G C,& quia numerus addendus ad 36, eſt L N M,& ideo quadratum G C cum eo quod fit ex G C duplicato in C B,ſeu ex G C in duplum C B,& eſt 12,numerus quadratorum priorum,ducam igitur 1 poſitionem,dimidium numeri q̄dratorum additorū,ſemper in numerum q̄dratorū priorū, & in ſe,& fient 1 q̄dratum p: 12 poſitionibus addenda ex alia parte, & etiam 2 poſitiones pro numero quadratorum, habemus igitur iterum ex communi animi ſententia, quantitates infraſcriptas, inuicem æquales,& utraq; habent radicem,prima ex regula tertia, ſed ſecunda quantitas ex ſuppoſito,igitur ducta prima parte trinomij in tertiam,fit quadratum dimidiæ partis ſecundæ

1 q̄d'q̄d.p: 2 poſ.p: 12. q̄d* p: 1 q̄d.p: 12 poſ. additi numeri p: 36 æqualia.
2 poſ.6 q̄dratorū,p: 60 poſ.p: 1 q̄d. p: 12 poſ. numeri additi.

trinomij,quia igitur ex dimidio ſecundæ in ſe,fiunt 900,quadrata,& ex prima in tertiam, fiunt 2 cubi p: 30 quadratis p: 72 poſitionibus quadratorum,ſimiliter erit deprimendo per quadrata, quia æqualia per æqualia diuiſa,producunt æqualia,ut 2 cu. p: 30 quadratis p: 72 poſitionibus æquantur 900,quare 1 cubus p: 15 quadratis p: 36 poſitionibus æquantur 450.

Sufficit igitur deducendo ad regulam,habere ſemper 1 cubum p: numero priorum quadratorum,addita ei quarta parte p: numero poſitionum tali,qualis eſt numerus æquationis primus,ut ſi haberimus 1 q̄d'q̄dratum p: 12 quadratis p: 36, æqualia 6 quadratis p: 60 poſitionibus,habebimus 1 cubum p: 15 quadratis p: 36 poſitionibus æqualia 450,dimidio quadrati dimidij numeri poſitionum, & ſi haberemus 1 q̄d'q̄dratum p: 16 quadratis p: 64 æqualia 80 poſitionibus, haberemus 1 cubum p: 20 quadratis p: 64 poſitionibus æqualia 800, & ſi haberemus 1 q̄d'q̄dratum p: 20 quadratis p: 100,æqualia 80 poſitionibus,haberemus 1 cubum p: 25 quadratis p: 100 poſitionibus æqualia 800,igitur hoc habito,in priore exemplo habuimus, 1 cub. p: 15 quadratis p: 36 poſitionibus æqualia 450, igitur rei æſtimatio, per 17m capitulum,eſt ℞ v: cubica 287 1/2 p: ℞ 80449 1/4, p: ℞ v: cubica 287 1/2, m: ℞ 80449 1/4 m: 5, hic igitur eſt numerus quadratorū,qui duplicatus, eſt addendus ex utraq; parte,quia ſupponuntur 2 res addendæ,& numerus addendus ex utraq; parte,ex demonſtratione, eſt quadratum huius,cum eo quod fit ex hoc in 12,numerum quadratorum,

T 2

Fig. 2