This three then expresses one root of the square figure, that is, one root of the proposed square of the unknown, and 9 the square itself. Hence we take half of ten and multiply this by itself. We then add the whole product of the multiplication to 39, that the drawing of the larger square GH may be completed; for the lack of the four corners rendered incomplete the drawing of the whole of this square. Now it is evident that the fourth part of any number multiplied by itself and then multiplied by four gives the same number as half of the number multiplied by itself. Therefore if half of the root is multiplied by itself, the sum total of this multiplication will wipe out, equal, or cancel the multiplication of the fourth part by itself and then by four.

Fig. 2
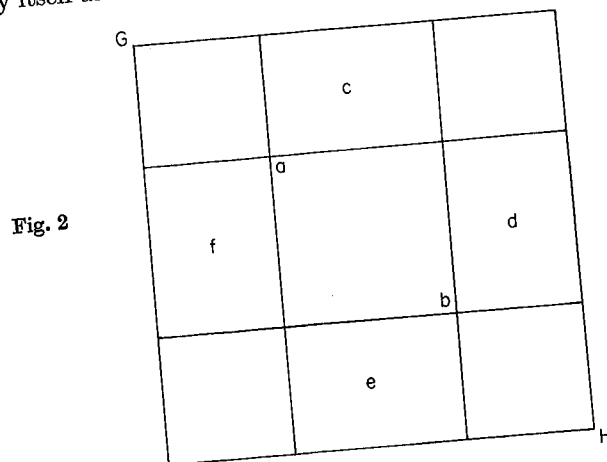
The remainder of the treatise deals with problems that can be reduced to one of the six types, for example, how to divide 10 into two parts in such a way that the sum of the products obtained by multiplying each part by itself is equal to 58: $x^2 + (10 - x)^2 = 58$, $x = 3, x = 7$. This is followed by a section on problems of inheritance.

2 CHUQUET. THE TRIPARTY

Nicolas Chuquet of Paris worked in Lyons, where he may have practiced medicine. His extensive work, *Le Triparty en la science des nombres du Maistre Nicolas Chuquet Parisien* (1484), so called because the book is divided into three sections (computation with rational numbers, computation with irrational numbers, and theory of equations), was not printed until 1880, but had considerable influence in manuscript. The book shows that in the mercantile city of Lyons a good deal of arithmetic and algebra was known, comparable to that dealing cities of Italy and Germany. It can be studied in the *Bollettino di Biblio*... *Fisiche 13* (1880), 593–814, edited by B...

notation, which uses special hieroglyphs for what we write (with Descartes) as $x, x^2, x^3, x^4, \ldots$ (see, for example, the reproduced page of Recorde in Smith, *History of mathematics*, II, 412; also 427–431). In Chuquet we find 12^1 for our $12x$, 12^2 for $12x^2$, 12^3 for $12x^3$, and so on, and this consistently, so that $12^0 = 12$ and $12^{-1} = 12/x$ (-1 is written $1.\tilde{m}$); these exponents are called "denominacions" (this use of negative numbers was quite unusual in those days). Chuquet then shows that $x^m \cdot x^n = x^{m+n}$. How Chuquet does it we see from the following translation of a part of the fourth chapter (pp. 739–740, 746):

How to multiply a difference of number [*une differance de nombre*] by itself or by another similar or dissimilar to it.

Example. He who multiplies $.12^0_{\cdot}$ by $.12^0_{\cdot}$ obtains $.144.$, then he who adds $.0.$ to $.0.$ obtains $.0.$; hence this multiplication gives $.144..$[1]

This means $12x^0 \cdot 12x^0 = 144x^{0+0} = 144x^0$.

Then he who multiplies $.12^0_{\cdot}$ by $.10^2_{\cdot}$ has first to multiply $.12.$ by $.10.$, which gives $.120.$, and then $.0.$ must be added to $.2..$ Thus the multiplication will give $120^2_{\cdot}$. By the same reasoning he who multiplies $.5^1_{\cdot}$ by $.8^1_{\cdot}$ obtains the multiplication $.40^2_{\cdot}$.

He who also wants to multiply $.12^3_{\cdot}$ by $.10^5_{\cdot}$ must first multiply $.12.$ by $.10.$, obtaining $.120.$, then must add the denominations together, which are $.3.$ and $.5.$, giving $.8..$ Hence the multiplication gives $.120^8_{\cdot}$.

Also he who wants to multiply $.8^1_{\cdot}$ by $.7^{1.\tilde{m}}_{\cdot}$ obtains as multiplication $.56.$, then he who adds the denominations together will take $1.\tilde{p}$ with $.1.\tilde{m}$ and obtains $.0..$ Here he obtains the multiplication $.56^0_{\cdot}$.[2]

Similarly, he who would multiply $.8^3_{\cdot}$ by $.7^{1.\tilde{m}}_{\cdot}$ will find it convenient first to multiply $.8.$ by $.7.$. He obtains $.56.$, then he must add the denominations, and will take $3.\tilde{p}$ with $.1.\tilde{m}$ and obtain $.2..$ Hence the multiplication gives $.56^2_{\cdot}$ and in this way we must understand other problems.

Then follows a list of powers of 2 with their denominations:

number	1	2	4	8	16	32	64	$\cdots$	1048576
denomination	0	1	2	3	4	5	6	$\cdots$	20

ON EQUIPOLLENCES OF NUMBERS

... And in order better to understand what has been said above about this art and style of abbreviating and equating its terms [*parties*] and of bringing them back to two simple terms as well as one can do it, we shall give here some examples of which the first is as follows—I shall abbreviate:

$$\mathrm{R}^2 \; \underline{4.^2\tilde{p}.4^1} \; \tilde{p}.2^{\frac12}\tilde{p}.1 \quad \text{equal to } .100.^3$$

First I take away $.2^{\frac12}\tilde{p}.1$ from both terms and there remains to me $\mathrm{R}^2 \underline{4.^2\tilde{p}.4^1}$ in one term and $99.\tilde{m}.2^{\frac12}$ in the other. And now that one of the terms is a second in this root it is convenient to multiply it by itself and we obtain $.4.^2.1.\tilde{p}.4^1$ in this term. And similarly we must multiply $.99.\tilde{m}.2^{\frac12}$ by itself and we obtain $9801.\tilde{m}.396^{\frac12}\tilde{p}.4.^2$ in the other term. Now we still must abbreviate these terms by taking away $.4.^2$ from the one and the other term. And then add $.396^{\frac12}$ to each of them. In this way we shall have $.400^{\frac12}$ in one term and $.9801.$ in the other term.[4]

After more of this there follows a theory of quadratic equations, in which negative roots are rejected. Chuquet has negative but no fractional exponents. Those we meet, even before Chuquet, in the *Algorismus proportionum* by Nicole Oresme (c. 1323–1382; see Selection III.1). Here we find a notation $\boxed{\begin{matrix}1\\2\end{matrix}}$ for $\frac12$, $\boxed{\begin{matrix}p & 1.\\1 & .2.\end{matrix}}$ for $1\frac12$, $\boxed{\begin{matrix}p & 1\\1 & 3\end{matrix}}$ for $1\frac13$, $\boxed{\begin{matrix}p & 2\\2 & 4\end{matrix}}$ for $2\frac24$, and $\boxed{\begin{matrix}p & 1.\\1 & 2.\end{matrix}}$ 4 for $4^{1\frac12}$. The p stands for *proportio*. The dots (.) are sometimes present, sometimes absent in the manuscript text reproduced by F. Cajori, *History of mathematical notations* (Open Court, Chicago, 1928), I, 92. A variant $\boxed{1^{p\frac12}}$ 4 is found in Cantor, *Geschichte*, II, 121. On the *Algorismus* see *De proportionibus proportionum and Ad pauca respicientes*, ed. E. Grant (University of Wisconsin Press, Madison, 1966), 65–68.

3 CARDAN. ON CUBIC EQUATIONS

The discovery of the numerical solutions of equations of the third degree at the University of Bologna in the early years of the sixteenth century was an important step in the development of algebra. It attracted wide attention, and was discussed in many public disputations. The textbook that laid the whole method open to public inspection was the *Ars magna* (Nuremberg, 1545) by the physician, humanist, mathematician, and scientist-in-general Gerolamo Cardano, or Hieronymus Cardanus, or, in English, Jerome Cardan (1501–1576). It stated that Scipio del Ferro at Bologna had discovered the method of solving

and then found a method of solving equations of the type $x^3 = px + q$, $x^3 + q = px$. Cardan obtained the solutions from Tartaglia (breaking a pledge of secrecy) and the method of solving cubic equations numerically has ever since been called after Cardan. The *Ars magna* was for many decades the best-known book on algebra, studied by all who were interested, and it lost this position only when Descartes introduced his new methods.

We quote here an English translation of a part of Chapter XI (pp. 29r–30r), dealing with the equation $x^3 + px = q$, or in particular $x^3 + 6x = 20$. It is based, as is also the text of Selection II.4, on the translation published in Smith, *Source book*, 204–212. Cardan's notation is quite different from ours, and he expresses the equation by saying: "A cube and unknowns are equal to a number" (Cubus et res aequales numero). For "unknown," our x, he has, like most of his contemporaries, the Latin term *res*, Italian *cosa*, literally, "thing." A cube is conceived as a solid body. By "number" is meant a numerical coefficient, in this case 20.

The book contains solutions for quadratics and for many types of cubes and biquadratics. The coefficients are always positive and specific numbers. Cardan also teaches some properties of equations and their roots. For instance (in Chapter XVII) we read that the equation $x^3 + 10x = 6x^2 + 4$ has three roots, namely 2, $2 + \sqrt2$, $2 - \sqrt2$, and Cardan sees that their sum adds up to the coefficient of x^2. Cardan is puzzled when imaginaries appear, and keeps them out of the *Ars magna* except in one case (see below), where he meets them in the solution of a quadratic equation. The *casus irreducibilis*, where a real root appears as a sum of the cube roots of two imaginaries (as in $x^3 = 15x + 4$, where $x = 4$, but the Cardan formula gives $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$) is discussed in the works of Bombelli (1572) and Viète (1591).

On Cardan see O. Ore, *Cardano, the gambling scholar* (Princeton University Press, Princeton, New Jersey, 1953). On the *Ars magna* see J. F. Scott, *A history of mathematics* (Taylor and Francis, London, 1958), 87–92. On Italian mathematicians of the Renaissance, see further E. Bortolotti, *Studi e ricerche sulla storia della matematica in Italia nei secoli XVI e XVII* (Zanichelli, Bologna, 1928).

CONCERNING A CUBE AND UNKNOWNS EQUAL TO A NUMBER

Chapter XI

Scipio del Ferro of Bologna about thirty years ago invented [the method set forth in] this chapter, [and] communicated it to Antonio Maria Florido of Venice, who when he once engaged in a contest with Nicolo Tartaglia of Brescia announced that Nicolo also invented it: and he [Nicolo] communicated it to us when we asked for it, but suppressed the demonstration.[1] With this aid we sought the demonstration, and found it, though with great difficulty, in the manner which we set out in the following.

Demonstration. For example, let the cube of *GH* and six times the side *GH* be

[1] Tartaglia and Cardan met in Milan during 1539, after which Tartaglia gave Cardan his method in obscure verses, which he later clarified. Cardan soon mastered the method and knew how to apply it independently. The verses begin as follows:

Quando che'l cubo con le cose appresso   When x^3 together with px