Fundamental Theorems of Algebra for the Perplexes

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doi:10.4169/074683409X475643

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The *perplex numbers* $\mathbb{P}$ (also called the *hyperbolic numbers* **[6, 7]**, the *spacetime numbers* **[2, 3]**, and sometimes the *split-complex numbers* **[8]**) are, like the complexes, a two-dimensional number system over the reals. Every perplex number z has the form $z = t + xh$, where t and x are real numbers. But h, rather than being a square root of minus 1, is a square root of *plus* 1, an extra such root, supplementing ± 1, the preexisting, well-known, customary and usual, real square roots of 1.

The perplex numbers tend to be rediscovered every few years and put to various uses. They are related, for example, to the hyperbolic geometry Einstein used to define special relativity. As Sobczyk **[6]** argues, they should get more attention from mathematicians and, in particular, deserve to be taught to undergraduates.

In this article, we review the basic properties of the perplex numbers, and then state and prove a fundamental theorem of algebra for them. In fact, rather surprisingly, we have a whole series of fundamental theorems for this intriguing, relatively obscure number system.

Properties of the perplex numbers

If $z = t + xh$ is a perplex number, then t is called the *real part* of z and x is called the *hyperbolic part.* Alternatively, t is often referred to as the time component, and x the space component **[2]**. (Fjelstad **[5]** dubbed x the *hallucinatory* part.)

Given perplex numbers $z_1 = t_1 + x_1 h$ and $z_2 = t_2 + x_2 h$, we have the basic operations:

$$z_1 + z_2 = (t_1 + t_2) + (x_1 + x_2)h,$$
$$z_1 - z_2 = (t_1 - t_2) + (x_1 - x_2)h,$$
$$z_1 z_2 = (t_1 t_2 + x_1 x_2) + (t_2 x_1 + t_1 x_2)h.$$

For division, it is usually advantageous to rationalize the denominator. To do this, we need a *perplex conjugate*: $\overline{z} = t - xh$. Then we can write

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(t_1 + x_1 h)(t_2 - x_2 h)}{(t_2 + x_2 h)(t_2 - x_2 h)} = \frac{t_1 t_2 - x_1 x_2 + (t_2 x_1 - t_1 x_2)h}{t_2^2 - x_2^2}.$$

In particular, we see that for $z = t + xh$, its multiplicative inverse is

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{t - xh}{t^2 - x^2}.$$

As with the complex numbers, the quantity $z\overline{z} = t^2 - x^2$ is a real number; unlike the complex numbers, it can be negative or zero. We see that any perplex number $z = t + xh$ with $t = \pm x$ is a zero divisor. This means that while $\mathbb{C}$ is a field, the perplex numbers $\mathbb{P}$ are not even an integral domain, just a commutative ring.

We next define the *perplex modulus*, or absolute value, of $z = t + xh$ to be

$$|z|_{\mathbb{P}} = \sqrt{|z\overline{z}|} = \sqrt{|t^2 - x^2|}.$$

Note that $|z|_{\mathbb{P}} \geq 0$ for all $z \in \mathbb{P}$. From above, we note that $1/z$ exists for any perplex number z such that $|z|_{\mathbb{P}} \neq 0$, that is, for all $z \in \mathbb{P}$ but our zero divisors.

We can identify the number $z = t + xh$ in the perplex plane with the point or vector (t, x). (See Figure 1.) We can then think about a "unit circle" in our plane, a graph of the set of perplex numbers z such that $|z|_{\mathbb{P}} = 1$. This will be a pair of hyperbolae with intercepts on the horizontal t-axis at 1 and -1 and on the vertical x-axis at h and $-h$.

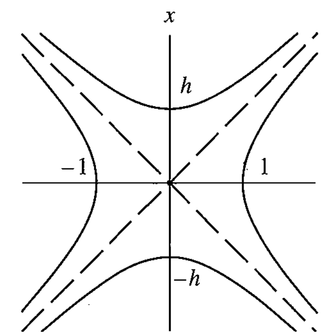

Figure 1. The perplex "unit circle." The dashed lines are the zero divisors.

The perplex plane is often thought of as a two-dimensional projection of four-dimensional spacetime.

Definition. For a perplex number $z = t + xh$:

- If $|t| > |x|$, then z is a time-like number.
- If $|t| < |x|$, then z is a space-like number.
- If $|t| = |x|$, then z is a light-like number.