Finally, if I have $z^2 = az - bb$, I make NL equal to $\frac{1}{2}a$ and LM equal to b as before; then [Fig. 4], instead of joining the points M and N, I draw MGR parallel to LN, and with N as a center describe a circle through L cutting MGR

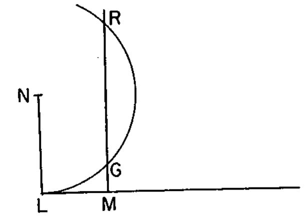

Fig. 4

in the points G and R; then z, the line sought, is either MG or MR, for in this case it can be expressed in two ways, namely,

$$z = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$$

and

$$z = \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}.$$

And if the circle described about N and passing through L neither cuts nor touches the line MGR, the equation has no root, so that we may say that the construction of the problem is impossible.[6]

These same roots can be found by many other methods. I have given these very simple ones to show that it is possible to construct all the problems of ordinary geometry by doing no more than the little covered in the four figures that I have explained. This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labor into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding them all, but rather gathered those propositions on which they had happened by accident.

Then Descartes takes up Pappus' problem (see Selection III.3) for the case of four lines. Here he introduces (Fig. 5) two segments AB and BC which he calls $AB = x$, $BC = y$. At this point he introduces what later would be called (oblique) coordinates. The four lines are AB, AD, EF, GH, to which at given angles are drawn from C the lines CB, CD, CF, and CH. He expresses the locus as an equation of the second degree. After some remarks on the case of more than four lines, he goes on to Book II (see Selection III.5).

[6] Descartes here follows the common practice of his day, which considered only the types $z^2 + az - b^2 = 0$, $z^2 - az - b^2 = 0$, and $z^2 - az + b^2 = 0$ of quadratic equations, ignoring the type $z^2 + az + b^2 = 0$, since it has no positive roots (a is a segment, hence positive). Only much later (Newton) did mathematicians begin to associate coordinates with negative numbers. All coordinates in Descartes are positive. The name "coordinate" does not appear in Descartes; this term is due to Leibniz.

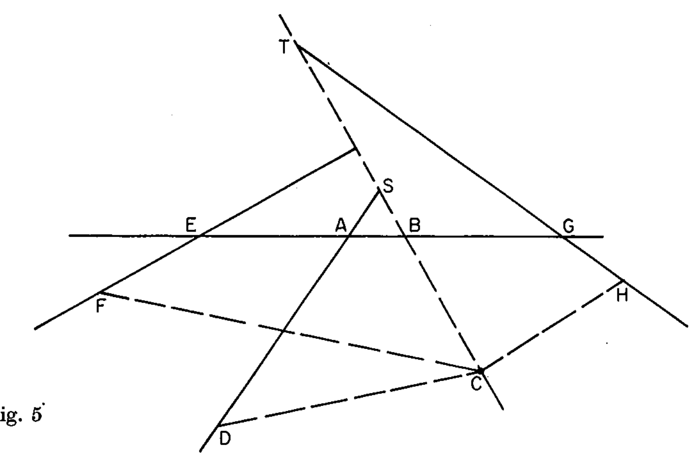

Fig. 5

5 DESCARTES. THE EQUATION OF A CURVE

In his *Géométrie* of 1637, Descartes applied his reformed algebra (see Selection II.7) to the geometry of the Ancients. In Book I he applies his coordinate method to Pappus' problem (see the previous Selection). The required locus can then be expressed by a relation between two variables which he denotes by x and y and in which we recognize oblique "Cartesian" corodinates.

"Since there is only one condition to be expressed ... we may give any value we please to either the one or the other of the unknown quantities x or y, and find the value of the other from this equation. It is evident that when no more than five lines are given, the quantity x, which is not used to express the first of the lines, can never be of degree [*dimension*] higher than the second. Assigning thus a given value to y, we have only $x^2 = \pm ax \pm b^2$ [*il ne restera que xx∞ + ou − ax + ou − bb*], and therefore the quantity x can be found with ruler and compasses, by a method already explained" (Smith and Latham, *The geometry of René Descartes*, p. 34; see Selection II.8).

Then, in Book II, after a classification of the problems of geometry into plane, solid, and linear ones (according to Pappus; see Selection III.2). Descartes suggests that a further classification of these "linear" curves is desirable, but that the classical distinction between geometrical and mechanical curves does not seem justified, since circles and straight lines can also be considered instruments [*machines*]. He then discusses some of these mechanical ways of describing a curve, and gives ((pp. 49–55 of the Smith-Latham translation) the following example of his coordinate method:

I wish to know the *genre*[1] of the curve EC [Fig. 1], which I imagine to be described by the intersection of the ruler GL and the rectilinear plane figure $CNKL$, whose side KN is produced indefinitely in the direction of C, and which,

[1] Earlier in Book II, Descartes has defined the *genre* of a curve. In our terms: If an algebraic curve has degree $2n - 1$ or $2n$, its *genre* is n. This terminology may have been inspired by the problem of Pappus. Newton (see Selection III.8) translates *genre* by *genus*.

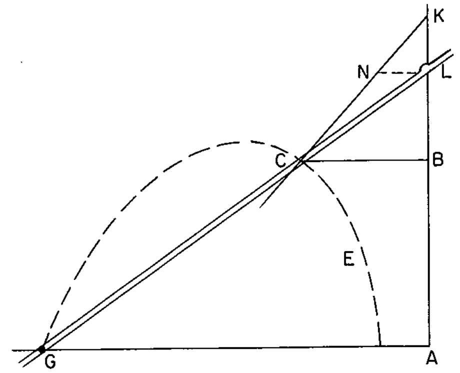

Fig. 1

being moved in the same plane in such a way that its side KL always coincides with some part of the line BA (produced in both directions), imparts to the ruler GL a rotary motion about G (the ruler being so connected to the figure $CNKL$ that it always passes through L).[2] If I wish to find out to what *genre* this curve belongs, I choose a straight line, as AB, to which to refer all its points, and in AB I choose a point like A at which to begin the calculation. I say that I choose the one and the other, because we are free to choose them as we like, for while it is necessary to use care in the choice in order to make the equation as short and simple as possible, yet no matter what line I should take instead of AB the curve would always prove to be of the same *genre*, a fact easily demonstrated.

Then I take on the curve an arbitrary point, as C, at which I will suppose that the instrument to describe the curve is applied. Then I draw through C the line CB parallel to GA. Since CB and BA are unknown and indeterminate quantities, I shall call one of them y and the other x. But in order to find the relation between these quantities I consider also the known quantities which determine the description of the curve, as GA, which I shall call a; KL, which I shall call b; and NL, parallel to GA, which I shall call c. Then I say that as NL is to LK, or as c is to b, so CB, or y, is to BK, which is therefore equal to $\frac{b}{c}y$. Then BL is equal to $\frac{b}{c}y - b$, and AL is equal to $x + \frac{b}{c}y - b$. Moreover, as CB is to LB, that is, as y is to $\frac{b}{c}y - b$, so AG or a is to LA or $x + \frac{b}{c}y - b$. Multiplying the second by the third, we get $\frac{ab}{c}y - ab$ equal to

$$xy + \frac{b}{c}yy - by,$$

[2] The instrument thus consists of three parts: (1) a ruler AK of indefinite length, fixed in the plane; (2) a ruler GL, also of indefinite length, passing through a pivot G in this plane (but not on AK); and (3) a triangle LNK, KN indefinitely extended toward KC, to which the ruler GL is connected at L so as to make the triangle slide with its side KL along AB.

which is obtained by multiplying the first by the last. Therefore, the required equation is

$$yy = cy - \frac{cx}{b}y + ay - ac.$$

From this equation we see that the curve EC belongs to the first *genre*, it being, in fact, a hyperbola.

If in the instrument used to describe the curve we substitute for the straight line CNK this hyperbola or some other curve of the first *genre* lying in the plane $CNKL$, the intersection of this curve with the ruler GL will describe, instead of the hyperbola EC, another curve, which will be of the second *genre*.

Thus, if CNK be a circle having its center at L, then we shall describe the first Conchoid of the Ancients,[3] while if we use a parabola having KB as diameter we shall describe the curve which, as I have already said, is the first and simplest of the curves required in the problem of Pappus, that is, the one which furnishes the solution when five lines are given in position.[4]

Then Descartes continues with his solution of the problem of Pappus, which leads him to the consideration of conic sections and other curves with several types of equations, such as

$$y^2 = 2y - xy + rx - x^2,$$

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy,$$

$$x^2 = ry - \frac{r}{q}y^3,$$

$$y^2 - by^2 - cdy + bcd + dxy = 0.$$

Here also is Descartes's method of finding the equation of a normal to a curve. This method was in a sense opposed to that of Fermat, whose method was based on finding first the equation of a tangent to a curve (see Selection IV.8) and thus came close to the idea of a derivative.

Book III of the *Géométrie* contains algebra; see Selection II.7.

6 DESARGUES. INVOLUTION AND PERSPECTIVE TRIANGLES

Girard Desargues (1593–1662) was an architect and military engineer, who lived at Lyons, and for some time also at Paris, where he met other mathematicians, including Descartes. His work in the field of perspective, in which he derived many sweeping generalizations, was

[3] Pappus mentions four types of conchoid (shell curves); the first is the one we still call a conchoid, in polar coordinates $r = a + b \sec \theta$. It is a curve of the third degree, therefore of the second *genre* of Descartes.

[4] This is also a curve of the second *genre*.