III CONTINUITY OF THE STRAIGHT LINE

Of the greatest importance, however, is the fact that in the straight line there are infinitely many points which correspond to no rational number. If the point $p$ corresponds to the rational number $a$, then as is well known, the length $dp$ is commensurable with the invariable unit of measure used in the construction, i.e., there exists a third length, a so-called common measure, of which these two lengths are integral multiples. But the ancient Greeks already knew and had demonstrated that there are lengths incommensurable with a given unit of length, e.g., the diagonal of the square whose side is the unit of length. If we lay off such a length from point $0$ upon the line we obtain an end-point which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: the straight line is infinitely richer in point-individuals than the domain $R$ of rational numbers in number-individuals.

If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument $R$ constructed by the creation of the rational numbers, be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line.

The previous considerations are so familiar and well known to all that many will regard their repetition quite superfluous. Still I regard this recapitulation as necessary to prepare properly for the main question. For, the way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes—which itself is nowhere carefully defined—and explains numbers as the result of measuring such a magnitude by another of the same kind. Instead of this I demand that arithmetic shall be developed out of itself. Such comparison with non-arithmetical notions has furnished the immediate occasion for the extension of the number-concept, and, in a general way, been granted (though this was certainly not the case in the introduction of complex numbers); but this surely is insufficient groundwork for introducing the foreign notions into arithmetic, the less so since there are infinitely many numbers, just as negative as fractional rational numbers are negative, and a new creation, and as the laws of arithmetic operate with these numbers may be reduced to the laws of operation with positive integers, so we must develop completely to define integral numbers by means of the rational numbers alone. The question only is how to do this.

The above comparison of the $R$ of rational numbers with a straight line has led to the recognition of existence of gaps, of a certain incompleteness or discontinuity of the former, while we ascribe to the straight line completeness, absence of gaps, or continuity. In what then does this continuity consist? Everything must depend on the answer to this question, and only through it shall we obtain a scientific basis for the investigation of all continuous domains. By vague remarks upon the broken connection in the smallest parts obviously nothing is gained; the problem is to indicate a precise characteristic of continuity that can serve as the basis for valid deductions. For a long time I pondered over this in vain, but finally I found what I was seeking. This discovery will, perhaps, be differently estimated by different people; the majority may find it in substance very commonplace. It consists of the following. In the preceding section attention was called to the fact that every point $p$ of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i.e., in the following principle: if all points of the straight line fall into classes such that every point of the first class falls to the left of every point of the second class, then there is one and only one point which lies at the division of all points into these severing of the straight line into two portions.

I thought I shall not err when I assert that every one will at once see the truth of this statement: the operating with these numbers may be reduced to the laws of operation with positive integers, so we must develop completely to define integral numbers by means of the rational numbers alone. The question only is how to do this.

IV CREATION OF IRRATIONAL NUMBERS

From the last remarks it is sufficiently obvious how the discontinuous domain of rational numbers may be rendered complete so as to form a continuous domain. In Section I it was pointed out that every rational number affects a separation of the system $R$ into two classes such that every number $a_1$ of the first class $A_1$ is less than every number $a_2$ of the second class $A_2$; the number $a$ of the second class $A_2$, or the least number of the class $A_2$, if there is any separation of the system $R$ into two classes, $A_1$, $A_2$, is given which possesses only this characteristic property that every number $a_1$, $a_2$ in $A_1$, is less than $a$, and every number $a_2$ in $A_2$, is greater than $a$.

We shall call such a separation a cut (Schnitt) and designate it by $(A_1, A_2)$. Can we then say that every rational number produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different? This is the property that either among the numbers of the first class there exist a greatest or among the numbers of the second class a least number. And conversely, if a cut possesses this property, then it is produced by this greatest or least rational number.

But it is easy to show that there exist
The square of every rational number less than 1, 0, and equal to 1 is less than 1.

If we consider the second class \(A_2\) of all rational numbers, then every positive rational number \(a\) is less than 1, and equal to 1.

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111. From Was sind und was sollen die Zahlen? (1888)*
(Simply Infinite Systems)
RICHARD DEDEKIND

CHAPTER VI

71. Definition. A system $N$ is said to be simply infinite when there exists a similar transformation $\phi$ of $N$ in itself such that $N$ appears as chain (44) of an element not contained in $\phi_N$. We call this element, which we shall denote in what follows by the symbol 1, the base-element of $N$ and say the simply infinite system $N$ is set in order (geordnet) by this transformation $\phi$. If we retain the earlier convenient symbols for transforms and chains (44) then the essence of a simply infinite system $N$ consists in the existence of a transformation $\phi$ of $N$ and an element 1 which satisfy the following conditions:

$\beta N = \gamma$,

$\gamma$ is the element 1 not contained in $N'$.

$\alpha$. The transformation $\phi$ is similar.

Obviously it follows from $\alpha$, $\beta$, $\gamma$ that every simply infinite system $N$ is actually an infinite system (44) because it is similar to a proper part $N'$ of itself...

(Definition of a Transformation of the Number-Series by Induction)

CHAPTER IX

126. Theorem of the definition by induction. If there is given an arbitrary (similar or dissimilar) transformation $\phi$ of a system $\Omega$ in itself, and besides a terminate element $w$ in $\Omega$, then there exists one and only one transformation $\psi$ of the number-series $N$ which satisfies the condition:

$\psi(0) = w$,

$\forall n \in N$, $\psi(n + 1) = n \psi(n)$. Where $n$ represents every number.

Proof. Since, if there actually exists such a transformation $\psi$, there is contained in it by (21) a transformation $\phi_n$ of the system $Z_n$ which satisfies the conditions I, II, III stated in (125), then because there exists one and only one such transformation $\phi_n$, must necessarily $\psi(0) = \phi_n(w)$.

Since the $\phi$ is completely determined it follows also that there can exist only one such transformation $\phi$ (see the closing remark in (130)). That conversely the transformation $\psi$ determined by (n) also satisfies our conditions I, II, III, follows easily from (n) with reference to the properties I, II and (p) shown in (125), which was to be proved.