What is Differential Geometry?

Differential Geometry is the study of Geometric Properties using Differential and Integral (though mostly differential) Calculus.

What are Geometric Properties?

Geometric Properties are properties that are solely of the geometric object, not of how it happens to appear in space. These are properties that do not change under congruence. A function is a congruence if it preserves distances. Length is the most natural and simple geometric property. Think of all the properties that are preserved in two congruent triangles, for example: the lengths of the sides, the area, the angles between the sides, the height, the number of sides, what rotation reflection symmetries the object has, &c. In contrast, consider the properties that are *not* preserved: the slopes of the sides, what the coordinates of the vertices are, orientation of the vertices (whether ABC is a counterclockwise or a clockwise sequence), &c.

Why would you want to use calculus to study geometry? To be able to study geometric objects that are not uniform. The geometry you learned in high school was fine for studying objects made of points, circles, lines, and planes but it's not so good for studying any other objects (that aren't made of those pieces). In other words, anything that is made out of curves that vary in their curvature. In differential geometry we tend to work with objects that don't have corners, so that we can take derivatives as we want, but we may work with objects with corners piece-by-piece, much like working with a polyhedron using the tools of high school geometry.

In our work in Part A we saw some of the things that would be true about a geometric object in which the sum of the angles of at least one triangle was less than π. We only studied such a object based on that assumption about a triangle. We didn't have a particular geometric object to study. (We also saw some of the things that are true about a geometric object in which the sum of the angles of at least one triangle was greater than π. That geometric object was very familiar, since we live on a sphere and have access to many balls as models.) One of the main historical goals of geometry is to find an object which satisfies these different axioms. That is one of our main goals, and we will realise it by the end of the course - one way or another.

Starting on Monday we began to consider curves, and in particular to consider curves in $\mathbb{R}^2$. One thing to measure about curves in $\mathbb{R}^2$ is their curvature. In fact, we can rephrase what is written above so that we see that differential geometry considers curves of non-constant curvature (i.e. not just lines and circles). Not only is curvature a valuable way to measure what happens locally in a curve, it also measures *completely* what happens in a curve. Soon - perhaps this week - we will prove that two curves are congruent (in the sense above) if and only if they have the same curvature. This says that curvature is precisely the thing to check if you want to know if two curves are congruent. It's a simple and quite powerful concept.

In Chapter 7 we will extend this idea to curves in space. We can still compute curvature for space curves, using the same formulae, but they no longer characterise congruence - i.e. there are pairs of space curves with the same curvature that are not congruent. This is because they are moving in the third dimension in different ways. We will introduce a concept called torsion to measure how the curves are changing in the third dimension. We will then prove the analogue of what we proved for curves in the plane: two curves are congruent if and only if they have the same curvature *and* torsion. At the end of chapter 7 we will explore briefly what congruences in $\mathbb{R}^n$ look like. We might wonder idly at that point what it would take for two curves in $\mathbb{R}^n$ to be congruent. We probably won't consider this too much, as we'll be more tempted by considering two-dimensional objects in $\mathbb{R}^3$ (and beyond) than considering one-dimensional objects in $\mathbb{R}^n$.

In Chapter 8 we will begin our study of two-dimensional surfaces. Much like we have begun curves switching between $\mathbb{R}^2$ and $\mathbb{R}^n$, we'll do the same with surfaces between those in $\mathbb{R}^3$ and those in $\mathbb{R}^n$. Since we're less familiar with surfaces than with curves, we'll spend most of chapter 8 just getting familiar with them. As an interlude after chapter 8, we'll use our new understanding of surfaces to consider an interesting digression of different maps of the sphere / Earth. In particular we'll note the (rather obvious) fact that there is no perfect planar map of the sphere. We'll then consider the different properties that different maps have that make them each useful for different purposes.

In Chapter 9 we will resume our study of surfaces, following back on the path that was set out by curves. In particular we will develop concepts of curvature of a surface. As we get to this point, our theorems will become more serious and not as easy to prove. It is for this reason that we devote all of chapter 10 to proving the analogue for surfaces that we proved for curves - what we need to know in order to show that two surfaces are congruent.

In Chapter 11 we go back to a topic that we have already discussed in Chapter 1. We discussed on the second day of the class that great circles, although they do not appear straight externally, are straight from the perspective within the sphere. In Chapter 11 we consider how to tell if a curve in a surface appears straight within a surface. Such a curve is called a geodesic. Perhaps unsurprisingly, no matter how hilly and bumpy a surface is, there are still ways to walk straight. This should be familiar - you could still drive straight without turning your car wheel (or ATV wheel) if you were driving through the mountains.

Chapter 12 is concerned with one of the most important and famous theorems in differential geometry. This is one of our main goals for the course - the Gauss (excuse the misspelling)-Bonnet Theorem. This theorem compares the local information of the curvature to global information of the entire surface. It's an elegant result that says if you put together all the pieces you can tell something about the whole. Our book has a particularly elegant presentation of this deep theorem which makes is transparent - so that you can easily understand why it *must* be true.

Following this point we will surely be late into the semester. At this point I will do what I can to establish what we can to pursue the original goal - of finding a surface which has the properties studied in part A - having a triangle with angle sum less than π *and* having all right angles congruent.

And that will be the end of our course. I hope this overview gives you some help in seeing what's happening and how it fits together, and I strongly hope that it will help you to attend to the details along the way in our journey. This is not easy material, but it's rewarding. I desperately need commitment from each of you to working with us to pursue these ideas. One thing I know for sure is that I cannot do this alone, without enthusiasm from you. Will you join me?