11.A8 Permissible and impermissible methods in geometry

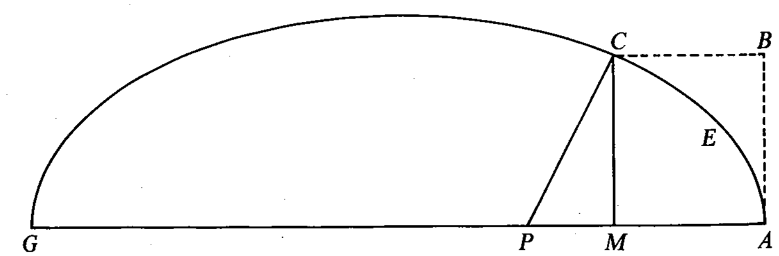
But the fact that this method of tracing a curve by determining a number of its points taken at random applies only to curves that can be generated by a regular and continuous motion does not justify its exclusion from geometry. Nor should we reject the method in which a string or loop of thread is used to determine the equality or difference of two or more straight lines drawn from each point of the required curve to certain other points, or making fixed angles with certain other lines. We have used this method in *La Dioptrique* in the discussion of the ellipse and the hyperbola.

On the other hand, geometry should not include lines that are like strings, in that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact. Nevertheless, since strings can be used in these constructions only to determine lines whose lengths are known, they need not be wholly excluded.

11.A9 The method of normals

[1] The angle formed by two intersecting curves can be as easily measured as the angle between two straight lines, provided that a straight line can be drawn making right angles with one of these curves at its point of intersection with the other. This is my reason for believing that I shall have given here a sufficient introduction to the study of curves when I have given a general method of drawing a straight line making right angles with a curve at an arbitrarily chosen point upon it. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know.

[2] Let CE be the curved line. It is desired to draw a straight line at right angles to it, through the point C. I suppose the problem to have been solved, and that the sought-for line is CP, which I prolong to the point P where it meets the straight line GA. (GA is the line to whose points all those of CE are referred; so that putting MA or CB equal to y, and CM or BA equal to x, I have some equation showing the relation between x and



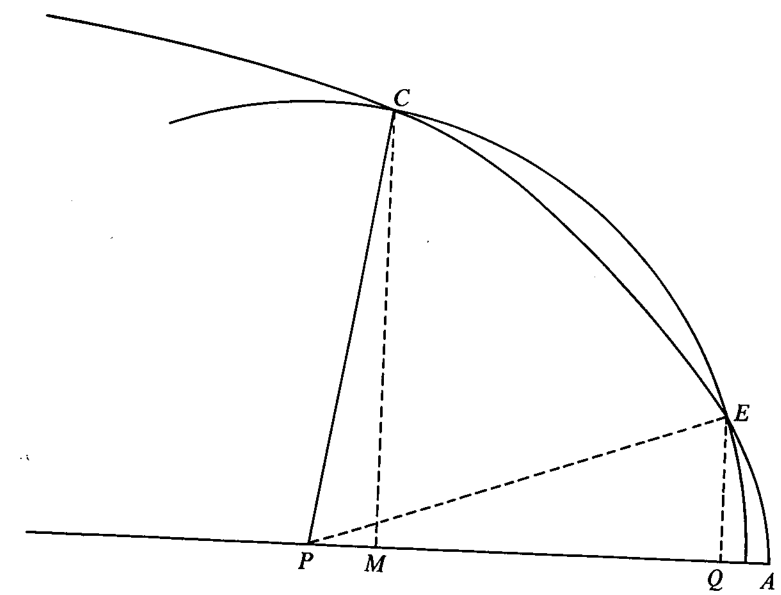
y.) Then I put $PC = s$, and $PA = v$, whence $PM = v - y$. Since the triangle PMC is right-angled, the square on the hypotenuse s^2 is equal to $x^2 + v^2 - 2vy + y^2$, the sum of the squares on the two sides. That is to say, $x = \sqrt{s^2 - v^2 + 2vy - y^2}$ or equally $y = v + \sqrt{s^2 - x^2}$. By this means I can get rid of one of the two unknown quantities x or y from the equation relating the points of the curve CE to those of the straight line GA. This is easily done by putting throughout $\sqrt{s^2 - v^2 + 2vy - y^2}$ in place of x, the square of this in the place of x^2, its cube in place of x^3, and so on. That is if it's x I want to get rid of; or if it's y, I put in its place $x + \sqrt{s^2 - x^2}$, and its square or cube, etc., in place of y^2, y^3 etc. After this process there always remains an equation in only one unknown quantity, x or y.

[3] For example, if CE is an Ellipse, MA the segment of its diameter on which CM is ordinate, and which has r for its *latus rectum* and q its major axis then by Book I Proposition 13 of Apollonius we have $x^2 = ry - ry^2/q$. Getting rid of x^2 from this gives $s^2 - v^2 + 2vy - y^2 = ry - ry^2/q$, or

$$y^2 + \frac{qry - 2qvy + qv^2 - qs^2}{q - r}$$

equals nothing. For it is better here to consider the whole together in this way, than as one part equal to the other. [. . .]

[4] Such an equation having been found it is to be used, not to determine x, y, or z, which are known, since the point C is given, but to find v or s, which determine the required point P. With this in view, observe that if the point P fulfills the required conditions, the circle about P as centre and passing through the point C will touch but not cut the curve CE; but if this point P be ever so little nearer to or farther from A than



it should be, this circle must cut the curve not only at C but also in another point. Now if this circle cuts CE, the equation involving x and y as unknown quantities (supposing PA and PC known) must have two unequal roots. Suppose, for example, that the circle cuts the curve in the points C and E. Draw EQ parallel to CM. Then x and y may be used to represent EQ and QA respectively in just the same way as they were used to represent CM and MA; since PE is equal to PC (being radii of the same circle), if we seek EQ and QA (supposing PE and PA given) we shall get the same equation that we should obtain by seeking CM and MA (supposing PC and PA given). It follows that the value of x, or y, or any other such quantity, will be two-fold in this equation, that is, the equation will have two unequal roots. If the value of x be required, one of these roots will be CM and the other EQ; while if y be required, one root will be MA and the other QA. It is true that if E is not on the same side of the curve as C, only one of these will be a true root, the other being drawn in the opposite direction, or less than nothing. The nearer together the points C and E are taken however, the less difference there is between the roots; and when the points coincide, the roots are exactly equal, that is to say, the circle through C will touch the curve CE at the point C without cutting it.

[5] Furthermore, it is to be observed that when an equation has two equal roots, its left-hand member must be similar in form to the expression obtained by multiplying by itself the difference between the unknown quantity and a known quantity equal to it; and then, if the resulting expression is not of as high a degree as the original equation, multiplying it by another expression which will make it of the same degree. This last step makes the two expressions correspond term by term.

[6] For example, I say that the first equation found in the present discussion, namely

$$y^2 + \frac{qry - 2qvy + qv^2 - qs^2}{q - r},$$

must be of the same form as the expression obtained by making $e = y$ and multiplying $y - e$ by itself, that is, as $y^2 - 2ey + e^2$. We may then compare the two expressions term by term, thus: Since the first term, y^2, is the same in each, the second term, $\dfrac{qry - 2qvy}{q - r}$, of the first is equal to $-2ey$, the second term of the second; whence, solving for v, or PA, we have $v = e - \dfrac{r}{q}e + \dfrac{1}{2}r$; or, since we have assumed e equal to y, $v = y - \dfrac{r}{q}y + \dfrac{1}{2}r$. In the same way, we can find s from the third term, $e^2 = \dfrac{qv^2 - qs^2}{q - r}$; but since v completely determines P, which is all that is required, it is not necessary to go further.

11.A10 H. J. M. Bos on Descartes's *Geometry*

Descartes's view can be summarized as follows: Construction of problems by ruler and compass is certainly simpler than, and therefore preferable to, construction by means of the intersection of conics or more complex curves. In the construction of problems one should always use the simplest possible curves. But this does not imply that more complex curves are necessarily less geometrical than the straight line and the circle, or that constructions by means of these curves are less geometrical than constructions by ruler and compass. There is a collection of curves of ever increasing complexity (circles, conics, conchoids, etc.) which are in principle acceptable in geometrical constructions. If a problem can be constructed by the intersection of two such curves and it cannot be constructed by simpler curves, then that construction is the right one to choose and it is no less geometrical a construction than one by ruler and compass.

This vision of the geometrical procedure of constructing problems determined a programme in three parts. First Descartes had to determine which curves were acceptable as genuinely geometrical means for the construction of problems. Secondly, he had to make it clear on which criteria some curves would be considered simpler than others: this would lead to a classification in order of simplicity within the collection of geometrically acceptable curves. Finally, a method had to be devised for finding the simplest possible curves by which each problem could be constructed. This is essentially the programme which Descartes worked out in his *Géométrie*.

The first point of the programme—differentiating between the curves which are acceptable in geometry and those which are not—caused Descartes (and his successors) the greatest number of conceptual problems. Basically Descartes took as geometrical curves those 'which can be described by some regular motion'. But this is not a very clear criterion. Also Descartes wished to include in the collection of geometrically acceptable curves all curves that may occur as locus solutions of problems such as the problem of Pappus. This meant that in fact—although Descartes never explicitly said so—he wanted to regard all algebraic curves as geometrical. But to do so he would have to prove that all algebraic curves could be traced by continuous and geometrically acceptable motions, or that they could be traced by other means which were just as geometrical as the tracing by continuous motion.

Algebra, in the sense of the existence of an algebraic equation of the curve, was the essential criterion in the first part of the programme. But the algebra had to remain implicit. Descartes could not simply take as 'geometrical' all curves that admit an algebraic equation, because obviously that is not a geometrical criterion; if he were to adopt this criterion, Descartes could no longer claim that he was doing geometry.

[...] As we have seen, the whole structure of his *Géométrie* depended on the conception of construction by the intersection of geometrical curves. For Descartes, these intersections were actually found or constructed only if the curves could be traced by continuous motion. In that case one can conceive clearly and distinctly that the intersections are found. If he were to renounce his criterion of tracing by continuous motion and at the same time keep to his programme of construction by the intersection of curves, he would have to state as an axiom that for all curves having an algebraic equation the intersections are given or constructible.

It is evident that Descartes could not do this. An axiom which states that the intersections of curves are constructible is by no means clearly and distinctly evident,