11.B4 Hendrik van Heuraet on the rectification of curves

Let there be given two curves, e.g. $ABCDE$, $GHIKL$, and the straight line AF such that (if from an arbitrary point M on line AF the perpendicular MI is drawn, cutting the given curves in C and I, while CQ is taken perpendicular to the curve $ABCDE$) MC is to CQ as a certain given segment Σ to MI: then the superficies $AGHIKLF$ will be equal to the rectangle contained by the given segment Σ and a segment equal to the curve $ABCDE$.

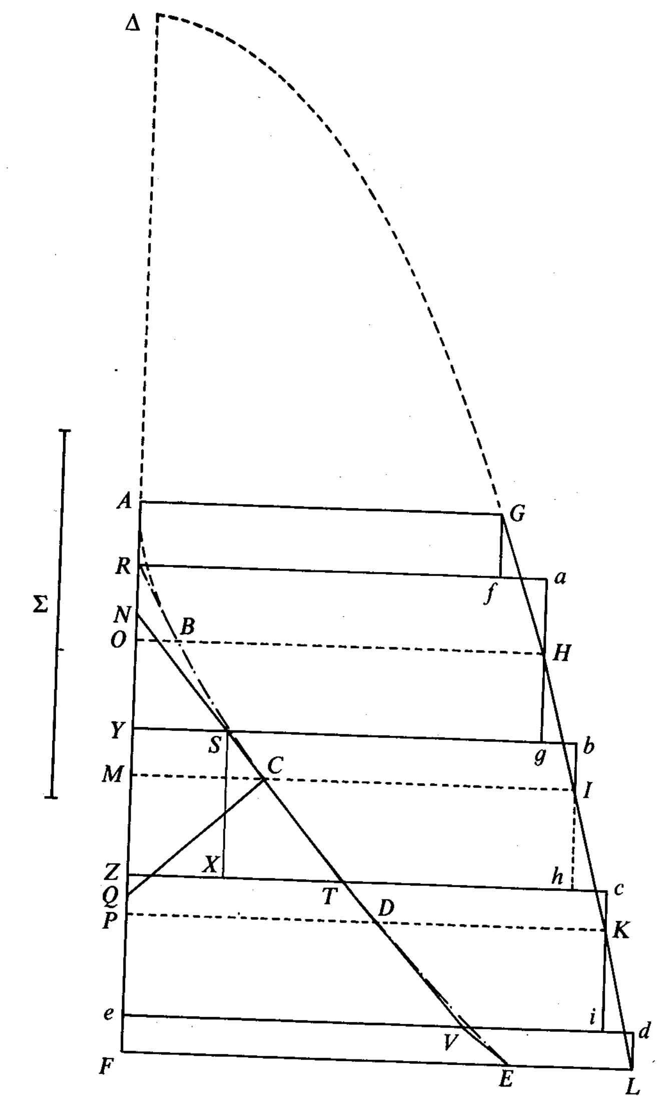
Let the line AF be divided in as many parts as you like, for example in the points O, M and P. And let the perpendiculars OH, MI, PK be drawn, cutting the curve $ABCDE$ in the points B, C and D, and the curve $GHIKL$ in the points H, I and K. And let through the points A, B, C, D, and E tangents be drawn, meeting each other in R, S, T, and V. And let through these points the lines Ra, Yb, Zc, ed be drawn, perpendicular to AF. And let through the points G, H, I, K and L lines be drawn parallel to AF, cutting Ra in f and a, Yb in g and b, Zc in h and c, ed in i and d. And finally let from S SX be drawn parallel to the line AF, and let the tangent TS be produced to N. Because of the right angle NCQ, CM will be to CQ as MN to NC. But MN is to NC as SX to ST. So SX will be to ST as CM to CQ. And because CM is to CQ as Σ to MI, there will also hold that SX is to ST as Σ to MI, and therefore the rectangle contained by SX or YZ and MI or Yb will be equal to the rectangle contained by S and Σ. In the same way one proves the rectangle ce to be equal to the rectangle contained by TV and Σ, and $\square dF \infty \square VE, \Sigma$ and $\square aY \infty \square$ contained by RS and Σ. Therefore these rectangles taken together will be equal to the rectangle contained by Σ and another segment, equal to the tangents taken together. Because this is true for any number of rectangles and tangents, and the figure consisting of the parallelograms will finally become the superficies $AGHIKLF$ if their number is increased to infinity, and because similarly the tangents will finally become the curve $ABCDE$, it is clear that the superficies $AGHIKLF$ is equal to the rectangle contained by Σ and another segment equal to the curve $ABCDE$. Which had to be proved.

However, how the length of the given curve can be investigated using this, will reveal itself in the following examples.

Let first the curve $ABCDE$ be such that, if on line AF an arbitrary point M is taken, if further the perpendicular MC is drawn, if AM is called x and MC is called y, yy always is $\infty \dfrac{x^3}{a}$. Further putting $AQ \infty s$, $CQ \infty v$ and $MI \infty z$; there will hold $QM \infty s - x$, and its square $\infty ss - 2sx + xx$. And if the square of MC, that is yy or $\dfrac{x^3}{a}$, is added to it, one will find $ss - 2sx + xx + \dfrac{x^3}{a} \infty vv$. Because of two equal roots, multiply according to Hudde's method with $\underline{0\quad 1\quad 2\quad 3\quad 0}$,

and one will find $\quad -2xs + 2xx + \dfrac{3x^3}{a} \infty 0.$

Therefore AQ or $s \infty x + \dfrac{3xx}{2a}$, and if from this $AM \infty x$ is subtracted, there will remain $MQ \infty \dfrac{3xx}{2a}$, the square of which is $\dfrac{9x^4}{4aa}$. To this add $\square CM$ or $\dfrac{x^3}{a}$, and there will appear



$\square CQ \infty \dfrac{9x^4}{4aa} + \dfrac{x^3}{a}$. Thus, as CM — i.e. $\sqrt{\dfrac{x^3}{a}}$ — will be to CQ — i.e. $\sqrt{\dfrac{9x^4}{4aa} + \dfrac{x^3}{a}}$ — so some known segment, put it $\frac{1}{3}a$ (for one may choose it arbitrary) to $MI \infty z$, and so you will have $z \infty \sqrt{\frac{1}{4}ax + \frac{1}{9}aa}$. And this argues that the line $GHIKL$ is a parabola, the vertex of which is in Δ, with $A\Delta \infty \frac{4}{9}a$, and with latus rectum $\infty \frac{1}{4}a$. And thus the length of the curve $ABCDE$ is $\sqrt{\dfrac{v^3}{a}} - \dfrac{8}{27}a$, if $\Delta F \infty v$. Similarly, if you put instead of $yy \infty \dfrac{x^3}{a}$

the following equation $y^4 \infty \frac{x^5}{a}$, or $y^6 \infty \frac{x^7}{a}$, or $y^8 \infty \frac{x^9}{a}$, and so on to infinity: you will always find such a superficies $AGHIKLF$ that it can be squared, and therefore all these curves can be transformed into a straight line. If, however, $ABCDE$ is a parabola, having axis AG and latus rectum $\infty\, a$: then you will find $MQ \infty \frac{2x^3}{aa}$ and its square $\infty \frac{4x^6}{a^4}$. Add to this CM squared, and you will have $\frac{4x^6}{a^4} + \frac{x^4}{aa}$ for $\square CQ$. From this: as CM—i.e. $\frac{xx}{a}$—is to CQ i.e. $\sqrt{\frac{4x^6}{a^4} + \frac{x^4}{aa}}$, so some known segment, put it a, to $MI \infty z$: and so you will have $z \infty \sqrt{4xx + aa}$, and the line $GHIKL$ will be a hyperbola, having axis AG, centre A, latus rectum $\infty \frac{1}{2}a$ and latus transversum $\infty\, 2a$. And from this exactly we learn that the length of the parabolic curve cannot be found unless at the same time the quadrature of the hyperbola is found, and vice versa.

11.B5 Jan Hudde's rules

If two roots of an equation are equal, and it is multiplied by an arithmetic progression as far as you like; to be sure, the first term of the equation by the first term of the progression, the second term of the equation by the second term of the progression and so on, I say that the product will be an equation in which one of the said roots reappears.

Hudde then first illustrated this for the polynomial

$$(x^3 + px^2 + qx + r)\ (x^2 - 2yx + y^2),$$

which plainly has the repeated root $x = y$, and the arithmetic progression a, $a \pm b$, $a \pm 2b$, etc. He showed that the resulting polynomial,

$$x^5a + (-2y + p)\ x^4(a + b) + (y^2 - 2py + q)x^3(a + 2b) + \text{etc.},$$

has $x = y$ as a root. To apply this observation to maxima and minima, Hudde argued that the repeated root of the original polynomial is now common to two polynomials, and can be found by a technique like the Euclidean algorithm, which finds the common factors of numbers.

11.C Pierre de Fermat

Pierre de Fermat (1601–1665) was a solitary man, never straying far from his native Toulouse, and hardly ever meeting a fellow mathematician. He preferred writing to individuals to any form of publication, and even when he felt he had something to say he was content to ask Marin Mersenne or, later, his friend Pierre de Carcavy to distribute copies of his letters. As a result much of what he wrote circulated in manuscript, if at all, in his lifetime, and was only published posthumously by his son Samuel in 1679. However, a large number of his letters survive and give a good picture of him as a mathematician. We have included as 11.C1–11.C2 some of his investigations into tangents and maxima and minima, where, like Descartes, but in the formalism of Viète (which he had learned as a young man and preferred), he gave an algebraic treatment of geometrical problems. Necessarily, the validity of his method rests on an appeal to geometric intuition—in this case in his appeal to an obscure concept of *adequality* (approximate equality) which he claimed to have learned from his reading of Diophantus. Indeed, Fermat's whole approach to the study of curves is rooted in his involvement in the contemporary enthusiasm for restoring lost Greek texts, and one is struck by how unsure he was all his life about the possibility of going far beyond the classical masters.

In 11.C5 we present his striking discovery of an area of finite size with an infinite perimeter. This is also one of the first times an area was found without the use of infinitesimals or indivisibles but by means of strips of finite width.

These achievements notwithstanding, perhaps Fermat's most profound insights were into the theory of numbers, a branch of mathematics in which he was to be without a follower, let alone a peer, until Leonhard Euler came on the scene. Inspired by Claude Gaspard Bachet's edition of Diophantus' *Arithmetica*, with its many ingenious problems, Fermat took up the question of finding all the solutions to a given question, or of showing that none could exist. His remarkable invention is the *method of descent*, which can be used on occasion to show that no solutions exist, and it is in this way that he despatched his case of what is nowadays called 'Fermat's last theorem' (see 11.C8). As is clear from 11.C7, Fermat was well aware of the power of his method; sadly he failed to interest Christiaan Huygens in it. Fermat's 'little theorem' (see 11.C4), although not at all difficult to prove once you have thought of it, derived from his interest in prime numbers, and we have included in this connection one of his few mistakes, 11.C3 on 'Fermat primes'. Much more profound is his grasp of 'Pell's equation', $x^2 = Ay^2 + 1$, a glimpse of which is given in 11.C6. From his choice of numbers it is clear that Fermat was in possession of a general theory for dealing with problems of this kind, for the solutions are far too large to be found by trial and error. For example, the smallest values of x and y for which $x^2 = 109y^2 + 1$ has solutions in integers are $x = 158070671986249$ and $y = 15140424455100$. André Weil, in his book cited in 11.C8(b), has argued convincingly that Fermat was in possession of a general method adequate for the problems he treated successfully, and so may be said to have had a theory of numbers. The reader is referred to this book, and to Weil's interesting essays, for a thorough analysis of Fermat as a number theorist.