That is why, having so to speak exhausted the properties of shaped extension through geometric speculations, we begin by restoring to it impenetrability, which constitutes physical body and was the last sensible quality of which we had divested it. The restoration of impenetrability brings with it the consideration of the action of bodies on one another, for bodies act only insofar as they are impenetrable. It thence that the laws of equilibrium and movement, which are the object of Mechanics, are deduced. We extend our investigations even to the movement of bodies animated by unknown driving forces or causes, provided the law whereby these causes act is known or supposed to be known.

Having at last made a complete return to the corporeal world, we soon perceive the use we can make of Geometry and Mechanics for acquiring the most varied and profound knowledge about the properties of bodies. It is approximately in this way that all the so-called physico-mathematical sciences were born. We can put at our head Astronomy, the study of which, next to the study of ourselves, is most worthy of our application because of the magnificent spectacle which it presents to us. Joining observation to calculation and elucidating the one by the other, this science determines with an admirable precision the distances and the most complicated movements of the heavenly bodies; it points out the very forces by which these movements are produced or altered. Thus it may justly be regarded as the most sublime and subtle application of Geometry and Mechanics in combination, and its progress may be considered the most incontestable monument of the success to which the human mind can rise by its efforts. [...] 

With respect to the mathematical sciences, which constitute the second of the limits of which we have spoken, their nature and their number should not overawe us. It is principally to the simplicity of their object that they owe their certitude. Indeed, one must confess that, since all the parts of mathematics do not have an equally simple aim, so also certainty, which is founded, properly speaking, on necessarily true and self-evident principles, does not belong equally or in the same way to all these parts. Several among them, supported by physical principles (that is, by truths of experience or by simple hypotheses), have, in a manner of speaking, only a certificate of experience or even pure supposition. To be specific, only those that deal with the calculation of magnitudes and with the general properties of extension, that is, Algebra, Geometry, and Mechanics, can be regarded as stamped by the seal of evidence. Indeed, there is a sort of gradation and shading, so to speak, to be observed in the enlightenment which these sciences bestow upon our minds. The broader the object they embrace and the more it is considered in a general and abstract manner, the more also their principles are exempt from obscurities. It is for this reason that Geometry is simpler than Mechanic, and both are less simple than Algebra. This will not be a paradox at all for whose who have studied these sciences philosophically. The most abstract notions, those that the common run of men regard as the most inaccessible, are often the ones which bring with them a greater illumination. Our ideas become increasingly obscure as we examine more and more sensible properties in an object. Impenetrability, added to the idea of extension, seems to offer us an additional mystery; the nature of movement is an enigma for the philosophers; the metaphysical principle of the laws of percussion is to less concealed from them. In a word, the more they delve into their conception of the matter and of the properties that represent it, the more this idea becomes obscure and seems to be trying to elude them.

[1] With regard to the solution of literal equations there has scarcely been any advance since the time of Cardano, who was the first to publish one for the equations of the third and fourth degrees. The first success of the Italian analysts in this subject seems to have been the end of the discoveries one can make there; at least it is certain that all the attempts which have been made to push back the limits of this part of algebra have still only served to find new methods for equations of the third and fourth degrees which do not appear to be applicable in general to equations of higher degree.

I propose in this memoir to examine the different methods which have been found up till now for the algebraic solution of equations, to reduce them to general principles, and to examine a priori why these methods succeed with the third and fourth degrees but fail for higher degrees.

This examination will have a two-fold advantage: on the one hand it will serve to shed greater light on the known solutions to the third and fourth degrees; on the other hand it will be useful to those who wish to occupy themselves with the solution of higher degrees, in providing them with different views of this object and above all in saving them a great number of steps and useless attempts. [...] 

[2] We conclude our analysis of the methods which concern the solution of equations of the fourth degree here. Not only have we related these methods to one another and shown their interconnections and their mutual dependence, but we have also, and this is the principal point, given the a priori reason why they lead, some to resolvents of the third degree, others to resolvents of the sixth, but which can be reduced to the third. One has seen how this derives in general from the fact that the roots of these resolvents are functions of quantities $x^3$, $x^4$, $x^5$, $x^6$, which, on making all the possible permutations of these four quantities, only receive three different values, like the function $x^3 + x^4$, or six values which of two are equal and of opposite sign, like the function $x^3 - x^7 - x^8$, or even six values which, on dividing them into three pairs and taking the sum or the product of the values of each pair, the three sums or the three products are always the same, whatever permutation one makes of the quantities $x$, $x^3$, $x^4$, $x^5$, $x^6$ [...] It is precisely the existence of such functions on which the solution of equations of the fourth degree depends. [...] 

[3] It follows from these reflections that it is very doubtful if the methods of which we have been speaking can give a complete solution of equations of the fifth degree, and still more so to those of higher degrees. And this uncertainty, coupled with the length of the calculations which these methods involve, must repel in advance all those who would seek to use them to solve one of the most famous and important problems in algebra. Also we observe that the authors of these methods have themselves been content to apply them to the third and fourth degrees and that no one has yet undertaken to push their work further.

It would therefore be very desirable if one could judge a priori the success that one can expect in applying these methods to degrees higher than the fourth. We are going to try and give the means for this by an analysis similar to that which has served us up till now in respect of the known methods for the solutions of equations of the third and fourth degree.
The geometricians of the last century paid great attention to the Indeterminate Analysis, or what is commonly called the Diophantine Algebra; but Bachet and Fermat alone can properly be said to have added anything to what Diophantus himself has left us on that subject.

To the former we particularly owe a complete method of resolving, in integer numbers, all indeterminate problems of the first degree: the latter is the author of some methods for the resolution of indeterminate equations, which exceed the second degree; of the singular method, by which we demonstrate that it is impossible for the sum, or the difference of two biquadrates to be a square; of the solution of a great number of very difficult problems; and of several admirable theorems respecting integer numbers, which he left without demonstration, but of which the greater part has since been demonstrated by M. Euler in the Petersburg Commentaries.

In the present century, this branch of analysis has been almost entirely neglected; and, except M. Euler, I know no person who has applied to it: but the beautiful and numerous discoveries, which that great mathematician has made in it, sufficiently compensate for the indifference which mathematical authors appear to have hitherto entertained for such researches. The Commentaries of Petersburg are full of the labours of M. Euler on this subject, and the preceding Work is a new service, which he has rendered to the admirers of the Diophantine Algebra. Before the publication of it, there was no work in which this science was treated methodically, and which enumerated and explained the principal rules hitherto known for the solution of indeterminate problems. The preceding Treatise unites both these advantages: but, in order to make it still more complete, I have thought it necessary to make several Additions to it, of which I shall now give a short account.

The theory of Continued Fractions is one of the most useful in arithmetic, as it serves to resolve problems with facility, which, without its aid, would be almost unmanageable; but it is of still greater utility in the solution of indeterminate problems, when integer numbers only are sought. This consideration has induced me to explain the theory of them, at sufficient length to make it understood. As it is not to be found in the chief works on arithmetic and algebra, it must be little known to mathematicians; and I shall be happy, if I can contribute to render it more familiar to them. At the end of this theory, which occupies the first Chapter, follow several curious and entirely new problems, depending on the truth of the same theory; but which I have thought proper to treat in a distinct manner, in order that the solution of them may become more interesting. Among these will be particularly remarked a very simple and easy method of reducing the roots of equations of the second degree to Continued Fractions, and a rigid demonstration, that those fractions must necessarily be always periodical.

The other Additions chiefly relate to the resolution of indeterminate equations of the first and second degree; for these I give new and general methods, both for the case in which the numbers are only required to be rational, and for that in which the numbers sought are required to be integer; and I consider some other important matters relating to the same subject.

The last Chapter contains researches on the functions, which have this property, that the product of two or more similar functions is always a similar function. I give a general method for finding such functions, and shew their use in the resolution of different indeterminate problems, to which the usual methods could not be applied. Such are the principal objects of these Additions, which might have been made much more extensive, had it not been for exceeding proper bounds; I hope, however, that the subjects here treated will merit the attention of mathematicians, and revive a taste for this branch of algebra, which appears to me very worthy of exercising their skill.

Chapter VIII
Remarks on Equations of the form $x^2 = Aq^2 + 1$, and on the common method of resolving them in Whole Numbers.

The method of Chap VII of the preceding Treatise, for resolving equations of this kind, is the same that Wallis gives in his Algebra (Chap. XCVIII), and ascribes to Lord Brouncker. We find it, also, in the Algebra of Ozanam, who gives the honour of it to M. Fermat. Whoever was the inventor of this method, it is at least certain, that M. Fermat was the author of the problem which is the subject of it. He had proposed it as a challenge to all the English mathematicians, as we learn from the Commercium Epistolicum of Wallis; which led Lord Brouncker to the invention of the method in question. But it does not appear that this author was fully apprised of the importance of the problem which he resolved. We find nothing on the subject, even in the writings of Fermat, which we possess, nor in any of the works of the last century, which treat of the Indeterminate Analysis. It is natural to suppose that Fermat, who was particularly engaged in the theory of integer numbers, concerning which he has left us some very excellent theorems, had been led to the problem in question by his researches on the general resolution of equations of the form,

$$x^2 = Aq^2 + B,$$

to which all quadratic equations of two unknown quantities are reducible. However, we are indebted to Euler alone for the remark, that this problem is necessary for finding all the possible solutions of such equations.

The method which I have pursued for demonstrating this proposition is somewhat different from that of M. Euler; but it is, if I am not mistaken, more direct and more general. For, on the one hand, the method of M. Euler naturally leads to fractional expressions, where it is required to avoid them; and, on the other, it does not appear very evidently, that the suppositions, which are made in order to remove the fractions, are the only ones that could have taken place. Indeed, we have elsewhere shewn, that the finding of one solution of the equation $x^2 = Aq^2 + B$, is not always sufficient to enable us to deduce others from it, by means of the equation $p^2 = Aq^2 + 1$; and that, frequently, at least when $B$ is not a prime number, there may be values of $x$ and $y$, which cannot be contained in the general expressions of M. Euler.

With regard to the manner of resolving equations of the form $p^2 = Aq^2 + 1$, I think that of Chap. VII, however ingenious it may be, is still far from being perfect. For, in the first place, it does not shew that every equation of this kind is always resolvable in whose numbers, when $A$ is a positive number not a square. Secondly, it is not demonstrated, that it must always lead to the solution sought for. Wallis, indeed, has professed to prove the former of these propositions; but his demonstration, if I may presume to say so, is a mere petitio principii (see Chap. XCIX). Mine, I believe, is the first rigid demonstration that has appeared. It is in the Mélanges de Turin, Vol. IV; but
1. Several conditions which an adequate map should satisfy have been recognized. It should: (1) not distort the shape of countries; (2) countries should maintain their true relative sizes; (3) the distance of every place from every other place should also be proportional to the true distance; (4) places lying on a straight line on the earth’s surface, that is, on a spherical great circle, should also lie on a straight line on the map; (5) the geographical latitude and longitude of places should be easily found on the map; and so on. In summary, a map should bear the same relation to countries, to hemispheres, or even to the entire earth, as do engineering drawings to a house, yard, garden, field or forest. This would work if the surface of the earth were a flat surface. But it is a spherical surface, and all of the requirements cannot be satisfied simultaneously, and it is therefore necessary to emphasize one or several especially valuable requirements at the expense of others. [...]

5. The notion that a terrestrial map should be viewed as a plan view of the earth’s surface has not been binding, and perspective sketches have been employed instead of plan views. Here the earth’s surface is drawn as it would be seen by the eye from a particular viewing point. [...]

6. [...] It is possible to choose innumerable points as the position of the eye from which to view the earth perspectively, but three advantageous points are emphasized. In one instance the eye is placed infinitely far from the globe, and this yields [...] orthographic projection. In a second instance the point is taken somewhere on the surface of the earth, and this method of projection is called stereographic, presumably because of the lack of a more specific expression. Finally, the eye is taken at the midpoint of the earth, and this method of projection, since no other name is known to me, will be called the central projection.

7. The central projection has the advantage that all spherical great circles are represented by straight lines. The small circles are in exceptional circumstances shown as circles but are otherwise always represented by conic sections. This method of composition therefore has the advantage that all places which lie on a straight line are on a great circle on the surface of the earth. I am not aware of any maps drawn in this manner, unless one includes those drawn on sundials where this construction occurs. Charts of the heavens, on the other hand, are advantageously drawn in this manner.

8. The three perspective compositions cited therefore have their advantages and disadvantages, and none satisfies all of the conditions cited in (1). In particular, the condition that the sizes of the countries maintain their true relations, is not found in any, and the conditions concerning measurement of separation of places either require restrictions or special constructions. [...]