I proceed to this subject in the following way. Let the ordinate be x, the abscissa y, and the interval between perpendicular and ordinate, described before, p. Then according to my method it follows immediately that $p\,dy = x\,dx$, as Dr. Craig has also found. When we now subject this differential equation to summation we obtain $\int p\,dy = \int x\,dx$ (like powers and roots in ordinary calculations, so here sum and difference, or $\int$ and d, are each other's converse). Hence we have $\int p\,dy = \frac{1}{2}xx$, which was to be demonstrated. Now I prefer to use dx and similar symbols rather than special letters, since this dx is a certain modification of the x and by virtue of this it happens that—when necessary—only the letter x with its powers and differentials enters into the calculus, and transcendental relations are expressed between x and some other quantity.[6] Transcendental curves can therefore also be expressed by an equation, for example, if a is an arc, and the versed sine x, then $a = \int dx : \sqrt{2x - x^2}$ and if the ordinate of a cycloid is y, then $y = \sqrt{2x - xx} + \int dx : \sqrt{2x - xx}$, which equation perfectly expresses the relation between the ordinate y and the abscissa x. From it all properties of the cycloid can be demonstrated. The analytic calculus is thus extended to those curves that hitherto have been excluded for no better reason than that they were thought to be unsuited to it. Wallis's interpolations and innumerable other questions can be derived from this.

3 LEIBNIZ. THE FUNDAMENTAL THEOREM OF THE CALCULUS

From the many papers that Leibniz wrote on the calculus we reproduce a part of the "Supplementum geometriae dimensoriae[1] . . . similiterque multiplex constructio lineae ex data tangentium conditione," *Acta Eruditorum* (1693), 385–392, translated from Leibniz, *Mathematische Schriften*, Abth. 2, Band I, 294–301. Here he expresses by means of a figure the inverse relation of integration and differentiation. There exists a German translation in Ostwald's *Klassiker*, No. 162 (Engelmann, Leipzig, 1908), 24–34.

I shall now show *that the general problem of quadratures can be reduced to the finding of a line that has a given law of tangency (declivitas)*, that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented. For this

[6] The meaning of this obscure Latin sentence seems to be (a) that the introduction of the dx under the integral sign makes it easier to see (what we would call) the functional character of the expression, as is clear when we pass from one variable to another and (b) that therefore we have a way of writing operating symbols expressing transcendental quantities such as $\int dx : x$, $\int dx : \sqrt{1 - x^2}$, symbols that express the nature of the quantity (contrary to such expressions as $\log x$ or $\sin x$, which in themselves do not express a property). Leibniz is often quite obscure when he wants to tell us about a really exciting discovery he has made.

[1] Leibniz distinguishes here between *geometria dimensoria*, which deals with quadratures, and *geometria determinatrix*, which can be reduced to algebraic equations.

purpose [Fig. 1] I assume for every curve $C(C')$ a *double characteristic triangle*,[2] one, TBC, that is assignable, and one, GLC, that is inassignable,[3] and these two are similar. The inassignable triangle consists of the parts GL, LC, with the elements of the coordinates CF, CB as sides, and GC, the element of arc, as the base or hypotenuse. But the assignable triangle TBC consists of the axis, the ordinate, and the tangent, and therefore contains the angle between the direction of the curve (or its tangent) and the axis or base, that is, the inclination of

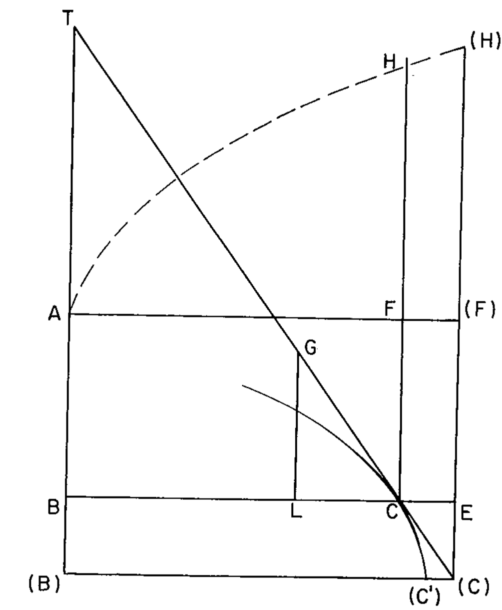

Fig. 1

the curve at the given point C. Now let $F(H)$, the region of which the area has to be squared,[4] be enclosed between the curve $H(H)$, the parallel lines FH and $(F)(H)$, and the axis $F(F)$; on that axis let A be a fixed point, and let a line AB, the conjugate axis, be drawn through A perpendicular to AF. We assume that point C lies on HF (continued if necessary); this gives a new curve $C(C')$ with the property that, if from point C to the conjugate axis AB (continued if necessary) both its ordinate CB (equal to AF) and tangent CT are drawn, the part TB of the axis between them is to BC as HF to a constant [segment] a, or a times BT is equal to the rectangle AFH (circumscribed about the trilinear figure

[2] In Fig. 1 Leibniz assigns the symbol (C) to two points which we denote by (C) and (C'). If, with Leibniz, we write $CF = x$, $BC = y$, $HF = z$, then $E(C) = dx$, $CE = F(F) = dy$, and $H(H)(F)F = z\,dy$. First Leibniz introduces curve $C(C')$ with its characteristic triangle, and then later reintroduces it as the squareing curve [*curva quadratrix*] of curve $AH(H)$.

[3] For want of anything better we use Leibniz's terms *assignabilis* and *inassignabilis*. G. Kowalewski, *Leibniz über die Analysis des Unendlichen*, Ostwald's *Klassiker*, No. 162 (Engelmann, Leipzig, 1908), 30, uses the German *angebbar* and *unangebbar*, "indicable" and "unindicable." For "differential" Leibniz in our text uses the term "element." Observe also the use of the term "coordinates" (Latin *coordinatae*).

[4] The Latin is here a little more expressive than the English. From the Latin *quadrare* we can derive *quadrans*, *quadrandus*, *quadratrix*, *quadratura*, which can be translated by "to square," "squaring," "to be squared," "squaring curve" or "quadratrix," and "quadrature."

$AFHA$).[5] This being established, I claim that the rectangle on a and $E(C)$ (we must discriminate between the ordinates FC and $(F)(C)$ of the curve) is equal to the region $F(H)$. When therefore I continue line $H(H)$ to A, the trilinear figure $AFHA$ of the figure to be squared is equal to the rectangle with the constant a and the ordinate FC of the squaring curve as sides. This follows immediately from our calculus. Let $AF = y$, $FH = z$, $BT = t$, and $FC = x$; then $t = zy : a$, according to our assumption; on the other hand, $t = y\,dx : dy$ because of the property of the tangents expressed in our calculus. Hence $a\,dx = z\,dy$ and therefore $ax = \int z\,dy = AFHA$. Hence the curve $C(C')$ is the quadratrix with respect to the curve $H(H)$, while the ordinate FC of $C(C')$, multiplied by the constant a, makes the rectangle equal to the area, or the sum of the ordinates $H(H)$ corresponding to the corresponding abscissas AF. Therefore, since $BT : AF = FH : a$ (by assumption), and the relation of this FH to AF (which expresses the nature of the figure to be squared) is given, the relation of BT to FH or to BC, as well as that of BT to TC, will be given, that is, the relation between the sides of triangle TBC.[6] Hence, all that is needed to be able to perform the quadratures and measurements is to be able to describe the curve $C(C')$ (which, as we have shown, is the quadratrix), when the relation between the sides of the assignable characteristic triangle TBC (that is, the law of inclination of the curve) is given.

Leibniz continues by describing an instrument that can perform this construction.

4 NEWTON AND GREGORY. BINOMIAL SERIES

Isaac Newton started to work on what is now called the calculus in 1664 under Barrow at Cambridge (Selection IV.14). One of his early sources was the Latin edition by F. van Schooten of the *Géométrie* of Descartes, which also had contributions to the infinitesimal calculus. Newton's first manuscript notes date from 1665. Here we see emerge his "pricked" letters, such as $\dot{x}$ for our dx/dt. Studying Wallis's *Arithmetica infinitorum* he also discovered the binomial series. Then, in 1669, having studied Nicolas Mercator's *Logarithmotechnia* (London, 1668) and James Gregory's *De vera circuli et hyperbolae quadratura* (Padua, 1667), he composed the manuscript later published as *De analysi per aequationes numero terminorum infinitas* (ed. W. Jones, London, 1711). Expanding on his fluxional methods, he wrote another text in 1671, entitled *Methodus fluxionum et serierum infinitorum*, first published, in English translation, as *The method of fluxions and infinite series*, ed. John Colson (London, 1736); the original was first published by Samuel Horsley in the *Opera omnia* (London, 1779–1785), under the title *Geometria analytica*.

Then, in 1676, in two letters to Henry Oldenburg, the secretary of the Royal Society and, like Mersenne at an earlier date, a man whose scientific contacts connected him with practically all who worked in the exact sciences, Newton presented some of his results, especially on the binomial series and on fluxions. The letters were destined for Leibniz, then in his early struggles for the discovery of his own calculus (see Selection V.1). After some time, Newton's attention was directed toward mechanics and astronomy; the result was the immortal *Philosophiae naturalis principia mathematica* (London, 1687; *Principia* for short), with its exposition of the planetary theory on the basis of the law of universal gravity. Newton did not explain his theory of fluxions in this book, preferring to give his proofs in classical geometric form as Huygens had done. However, some of his lemmas and propositions present, in carefully chosen language, a few products of his meditations on the calculus, and we reprint them here as Selections V.5, 6. Then, finally, in an attempt to collect his thoughts on fluxions, Newton produced in 1693 a manuscript that was eventually published as *Tractatus de quadratura curvarum* (London, 1704), which we have chosen for Selection V.7, being, as it seems, part of the last formulation that Newton gave to his theory of fluxions.

The *Analysis per aequationes*, the *Quadratura curvarum*, and the *Methodus fluxionum* have been republished, in their eighteenth-century English translations, by D. T. Whiteside, *The mathematical works of Isaac Newton* (Johnson Reprint Co., New York, London, 1964).

Our first selection of Newton's work gives essential parts of his two letters of 1676 to Oldenburg, dealing in the main with the binomial series. By applying Wallis's methods of interpolation and extrapolation to new problems, Newton had taken the concept of negative and fractional exponents from Wallis, and so had been able to generalize the binomial theorem, already known for a long time for positive integral exponents (see Selection I.5 on the Pascal triangle), to these more generalized exponents, by which a polynomial expression was changed into an infinite series. He then was able to show how a great many series that already existed in the literature could be regarded as special cases, either directly or by differentiation or integration.

Here follow the two letters from Newton to Oldenburg; they are taken from *The correspondence of Isaac Newton*, ed. H. W. Turnbull (Cambridge University Press, New York, 1959), vol. 1.

LETTER OF JUNE 13, 1676

Most worthy Sir,

Though the modesty of Mr. Leibniz, in the extracts from his letter which you have lately sent me, pays great tribute to our countrymen for a certain theory of infinite series, about which there now begins to be some talk, yet I have no doubt that he has discovered not only a method for reducing any quantities whatever to such series, as he asserts, but also various shortened forms, perhaps like our own, if not even better. Since, however, he very much wants to know what has been discovered in this subject by the English, and since I myself fell upon this theory some years ago, I have sent you some of those things which occurred to me in order to satisfy his wishes, at any rate in part.

Fractions are reduced to infinite series by division; and radical quantities by extraction of the roots, by carrying out those operations in the symbols just as

[5] This is Pascal's expression; see Selections IV.11, 12.

[6] This reasoning is still very much like that of Barrow, Gregory, and Torricelli, but because Leibniz possesses the converse relation $a\,dx = x\,dy \leftrightarrow \int a\,dx = \int x\,dy$ he needs only one demonstration, where Barrow needed two (Lecture X, 11; XI, 19; Selection IV.14).