If $AQ$ or $TR = z$, and $AC = f$, while $BC = a$; then, \[ \frac{AC}{BC} = \frac{f}{a} = \frac{TR}{z} = \frac{z^3}{x^2} \] and thus
\[ x = \frac{az}{f}. \]

If $dx$ is constant, then $\frac{dz}{y}$ is also constant. Hence $c \frac{dy}{y} = \frac{a}{f} y^3 dz$, or $c \frac{dy}{y^2} = \frac{a}{f} y dz$, and $\frac{dy}{y} = \frac{a}{f} y^2 dz$, therefore $\frac{y^2}{2} = \frac{a}{f} \int y^2 dz$. Hence we have both the area of the figure and the moment to a certain extent (for something must be added on account of the obliquity); also $cy \frac{dy}{y^2} = \frac{a}{f} y^2 dz$.

Also $c \frac{dy}{y} = \frac{a}{f} dz$, and hence, $c \frac{dy}{y} = \frac{a}{f} z$. Now, unless I am greatly mistaken, $\frac{dy}{y}$ is in our power. The whole matter reduces to this, we must find the curve in which the ordinate is such that it is equal to the differences of the ordinates divided by the abscissae, and then find the quadrature of that figure.

Figures of this kind, in which the ordinates are $dy/y$, $dy/y^2$, $dy/y^3$, are to be sought in the same way as I have obtained those whose ordinates are $y dy$, $y^2 dy$, etc. Now $w/a = dy/y$, and since $dy$ may be taken to be constant and equal to $f$, therefore the curve, in which $w/a = dy/y$, will give $w = af$, which would be a hyperbola. Hence the figure, in which $dy/y = z$, is a hyperbola, no matter how you express $y$, as well as $y$ is expressed by $\phi^2$ we have $dy = 2b$, and $2b\phi^2 = \frac{2}{\phi}$. Now, $c \frac{dy}{y} = \frac{a}{f} z$, and therefore $c \frac{dy}{a} = \frac{a}{f} z$, which thus appertains to a logarithm.

Thus we have solved all the problems on the inverse method of tangents, which occur in Volume 3 of the Correspondence of Descartes, of which he solved one himself; but the solution is not given; the other he tried to solve but could not, stating that it was an irregular line, which in any case was not in human power, nor in the power of the angels unless the art of describing it is determined by some other means.

### 13.3.3 The first publication of the calculus

A new method for maxima and minima as well as tangents, which is neither impeded by fractional nor irrational quantities, and a remarkable type of calculus for them.

Let an axis $AX$ [Figure 1] and several curves such as $XY$, $AY$, $XY$, $AY$, $Z$ be given, of which the ordinates $FX$, $XY$, $AX$, $Z$ are perpendicular to the axis, are called $x, y, z, w, t, s, x$ respectively. The segment $AX$, cut off from the axis is called $s$. Let the tangents be $VB$, $WC, YD, ZE$, intersecting the axis respectively at $B, C, D, E$. Now some straight line selected arbitrarily is called $dx$, and the line which is to $dx$ as $s$ (or $w$, or $y$, or $z$) is to $XB$ (or $XC, or XD, or YE$) is called $dt$ (or $dw, or dy, or dx$), or the difference of these $s$ (or $w$, or $y$, or $z$). Under these assumptions we have the following rules of the calculus.

![Figure 1](image_url)
relation of \( z \) to \( x \) is considered, then we can decide whether \( dz \) is a positive quantity or less than zero (or negative). When the latter occurs, then the tangent \( ZE \) is directed toward \( A \), but in the opposite direction, down from \( X \). This happens when the ordinates \( z \) decrease with increasing \( x \). And since the ordinates \( x \) sometimes increase and sometimes decrease, \( dx \) will sometimes be positive and sometimes be negative; in the first case the tangent \( yB \) is directed toward \( A \), in the latter it is directed in the opposite sense. None of these cases happens in the intermediate position at \( M \), at the moment when \( v \) neither increases nor decreases, but is stationary. Then \( dv = 0 \), and it does not matter whether the quantity is positive or negative, since \( s + v = -0 \). At this place \( v \), that is, the ordinate \( LM \), is maximum (or, when the convexity is turned to the axis, minimum), and the tangent to the curve at \( M \) is directed neither in the direction from \( X \) up to \( A \), to approach the axis, nor down to the other side, but is parallel to the axis. When \( dv \) is infinite with respect to \( dx \), then the tangent is perpendicular to the axis, that is, it is the ordinate itself. When \( dv = dx \), then the tangent makes half a right angle with the axis. When with increasing ordinates \( v \) its increments or differences \( dv \) also increase (that is, when \( dv \) is positive, \( dv \) the difference of the differences, is also positive, and when \( dv \) is negative, \( dv \) is also negative), then the curve turns toward the axis its convexity, in the other case its concavity. Where the increment is maximum or minimum, or where the increments from decreasing turn into increasing, or the opposite, there is a point of inflection. Here concavity and convexity are interchanged, provided the ordinates too do not turn from increasing into decreasing or the opposite, because then the concavity or convexity would remain. However, it is impossible that the increments continue to increase or decrease, but the ordinates turn from increasing into decreasing, or the opposite. Hence a point of inflection occurs when \( dv = 0 \) while neither \( v \) nor \( dx \) is zero. The problem of finding inflection therefore has not, like that of finding a maximum, two equal roots, but three. This all depends on the correct use of the signs.

Sometimes it is better to use ambiguous signs, as we have done with the division, before it is determined what the precise sign is. When with increasing \( x \) \( y \) increases (or decreases), then the ambiguous signs in \( u^m \) and \( d \) must be determined in such a way that this fraction is a positive (or negative) quantity. But \( \neq \) means the opposite of \( \neq \), so that when one is \( = \) the other is \( < \) or vice versa. There also may be several ambiguities in the same computation, which I distinguish by parentheses. For example, let \( u = \frac{y}{v} \), then we must write
\[
\left( \frac{\pm y \, dx}{yy} \right) + \left( \frac{\pm z \, dy}{zz} \right) = d[\pm \frac{w}{v}],
\]
so that the ambiguities in the different terms may not be confused. We must take notice that an ambiguous sign with itself gives +, with its opposite gives −, while with another ambiguous sign it forms a new ambiguity depending on both.

Powers. \( dx^2 = ax^{2-1} \, dx \); for example, \( dx^3 = 3x^2 \cdot dx \). \( \frac{1}{x} = -\frac{a \, dx}{x^{2-1}} \); for example, \( dx = 1 \), then \( dw = -\frac{3 \, dx}{x^2} \).

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expresses the relation between \( x \) and \( y \) or between \( AX \) and \( XY \), where \( a, b, c, e, f, g, h \) are given. We wish to draw from a point \( Y \) the line \( YD \) tangent to the curve, or to find the ratio of the line \( DX \) to the given line \( XY \). We shall write for short \( n = a + bx \), \( r = c - xx, q = ex + fxx, \) \( y = gy + yy, \) \( s = hh + lx + mxx \). We obtain \( x = y + npq + axr + yyr/s = 0 \), which we call the second equation. From our calculus it follows that

\[
d(x/y) = (\pm xdy/y = dx y^1/y,
\]

and equally that

\[
d(x/y) = (\pm xdy/y = dx y^1/y,
\]

All these differential quantities from \( dx(x/y) \) to \( dy(y/s) \) added together give 0, and thus produce a third equation, obtained from the terms of the second equation by substituting their differential quantities. Now \( dn = b dx \) and \( dp = -2x dx \), \( d = c dx + 2fx dx, \) \( dr = 2y dy \), and \( ds = i dx + 2mx dx \). When we substitute these values into the third equation we obtain a fourth equation, in which the only remaining differential quantities, namely \( dx, dy \), are all outside of the denominators and without restrictions. Each term is multiplied either by \( dx \) or by \( dy \), so that the law of homogeneity always holds with respect to these two quantities, however complicated the computation may be. From this we can always obtain the value of \( dx/dy \), the ratio of \( dx \) to \( dy \), or the ratio of the required \( DX \) to the given \( XY \). In our case this ratio will be

\[
F:1:y(y)(Qoqpe + 2h(x)q^1(2x) + p(x)q^1(2x) + (e(x)q^1(2x)).
\]

Now \( x \) and \( y \) are given since point \( Y \) is given. Also given are the values of \( a, p, q, s \) expressed in \( x \) and \( y \), which we wrote down above. Hence we have obtained what we required. Although this example is rather complicated we have presented it to show how the above-mentioned rules can be used even in a more difficult computation. Now it remains to show that their use in cases easier to grasp.

Let two points \( C \) and \( E \) [Figure 2] be given and a line \( SS \) in the same plane. It is required to find a point \( F \) on \( SS \) such that when \( E \) and \( C \) are connected with \( F \) the sum of the rectangle of \( CF \) and a given line \( h \) and the rectangle of \( FE \) and a given line \( r \) are as small as possible. In other words, if \( SS \) is a line separating two media, and \( h \) represents the density of the medium on the side of \( C \) (say water), \( r \) that of the medium on the side of \( E \) (say air), then we ask for the point \( F \) such that the path from \( C \) to \( E \) via \( F \) is the shortest possible. Let us assume that all such possible sums of rectangles, or all possible paths, are represented by the ordinates \( KV \) of curve \( WV \) perpendicular to the line \( GK \) [Figure 1]. We shall call these ordinates \( w \). Then it is required to find their minimum \( NM \). Since \( C \) and \( E \) [Figure 2] are given, their perpendiculars to \( SS \) are also given, namely \( CP \) (which we call \( c \)) and \( EQ \) (which we call \( e \)); moreover \( PQ \) (which we call \( p \)) is

\[ \text{given. We denote} \ QF = GN \text{ or } AX \text{ by} \ f, \text{ and } EF \text{ by } g. \text{ Then} \ FP = p - x, \ f = \sqrt{cc + pp - 2px + xx} \text{ or } = \sqrt{1} \text{ for short; } g = \sqrt{ee + xx} \text{ or } = \sqrt{m} \text{ for short. Hence}
\]

\[ w = h(x)^2 + r(x)^2.
\]

The differential equation (since \( dw = 0 \) in the case of a minimum) is, according to our calculation,

\[ 0 = \pm(x)(\delta y + x q + p(x)q + c(x)q + 2(x)(2x) + \delta x).
\]

But \( dl = 2(p - x) dx \); \( dm = q(x) dx \); hence

\[ h(x) \delta y = x q(x).
\]

When we now apply this to dioptrics, and take \( f \) and \( g \), that is, \( CF \) and \( EF \), equal to each other (since the refraction at the point \( F \) is the same no matter how long the line \( CF \) may be), then \( h(x) \delta y = x q(x) \), or \( h(x) \delta y = x \delta x \). \( \delta x \) and \( \delta y \) are in inverse ratio to \( r \) and \( h \), the densities of the medium in which the incidence and the refraction take place. However, this density is not to be understood with respect to us, but to the resistance which the light rays meet. Thus we have a demonstration of the construction exhibited elsewhere in these Acta, where we presented a general foundation of optics, catoptrics, and dioptrics. Other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic.

This I shall explain by still another example. Let \( AD \) [Figure 3] be a curve of such a nature that, if we draw from one of its points, such as \( A \), lines \( 54, 35, 36, 37, 38, 39 \) to six fixed points \( 4, 5, 6, 7, 8, 9 \) on the axis, then their sum is equal to a given line. Let