If AQ or $TR = z$, and $AC = f$, while $BC = a$; then, $\frac{AC}{BC} = \frac{f}{a} = \frac{TR}{BR} = \frac{z}{x}$; and thus $x = \frac{az}{f}$.

If $\overline{dx}$ is constant, then $\overline{dz}$ is also constant. Hence $c\,dy = \frac{a}{f} y\,\overline{dz}$, or $cy = \frac{a}{f}\int \overline{y\,\overline{dz}}$, and $cy\,\overline{dy} = \frac{a}{f} y^2\,\overline{dz}$, therefore $c\frac{y^2}{2} = \frac{a}{f}\int y^2\,\overline{dz}$. Hence we have both the area of the figure and the moment to a certain extent (for something must be added on account of the obliquity); also $cz\,\overline{dy} = \frac{a}{f} yz\,\overline{dz}$, and therefore $c\int z\,\overline{dy} = \frac{a}{f}\int \overline{yz\,\overline{dz}}$.

Also $\frac{c\,\overline{dy}}{y} = \frac{a}{f} dz$, and hence, $c\int\frac{\overline{dy}}{y} = \frac{a}{f} z$. Now, unless I am greatly mistaken, $\int\frac{\overline{dy}}{y}$ is in our power. The whole matter reduces to this, we must find the curve in which the ordinate is such that it is equal to the differences of the ordinates divided by the abscissae, and then find the quadrature of that figure. $\overline{d\sqrt{ay}} = \frac{1}{\sqrt{ay}}$.

Figures of this kind, in which the ordinates are dy/y, dy/y^2, dy/y^3, are to be sought in the same way as I have obtained those whose ordinates are $y\,dy$, $y^2\,\overline{dy}$, etc. Now $w/a = \overline{dy}/y$, and since $\overline{dy}$ may be taken to be constant and equal to β, therefore the curve, in which $w/a = \overline{dy}/y$, will give $wy = \alpha\beta$, which would be a hyperbola. Hence the figure, in which $dy/y = z$, is a hyperbola, no matter how you express y, and if y is expressed by ϕ^2 we have $dy = 2\phi$, and $\frac{2\phi}{\phi^2} = \frac{2}{\phi}$. Now, $c\int\frac{dy}{y} = \frac{a}{f}z$, and therefore $\frac{fc}{a}\int\frac{dy}{y} = z$, which thus appertains to a logarithm.

Thus we have solved all the problems on the inverse method of tangents, which occur in Volume 3 of the *Correspondence* of Descartes, of which he solved one himself; but the solution is not given; the other he tried to solve but could not, stating that it was an irregular line, which in any case was not in human power, nay not within the power of the angels unless the art of describing it is determined by some other means.

13.A3 The first publication of the calculus

A new method for maxima and minima as well as tangents, which is neither impeded by fractional nor irrational quantities, and a remarkable type of calculus for them

Let an axis AX [Figure 1] and several curves such as VV, WW, YY, ZZ be given, of which the ordinates VX, WX, YX, ZX, perpendicular to the axis, are called v, w, y, z respectively. The segment AX, cut off from the axis is called x. Let the tangents be VB, WC, YD, ZE, intersecting the axis respectively at B, C, D, E. Now some straight line selected arbitrarily is called dx, and the line which is to dx as v (or w, or y, or z) is to XB (or XC, or XD, or XE) is called dv (or dw, or dy, or dz), or the difference of these v (or w, or y, or z). Under these assumptions we have the following rules of the calculus.

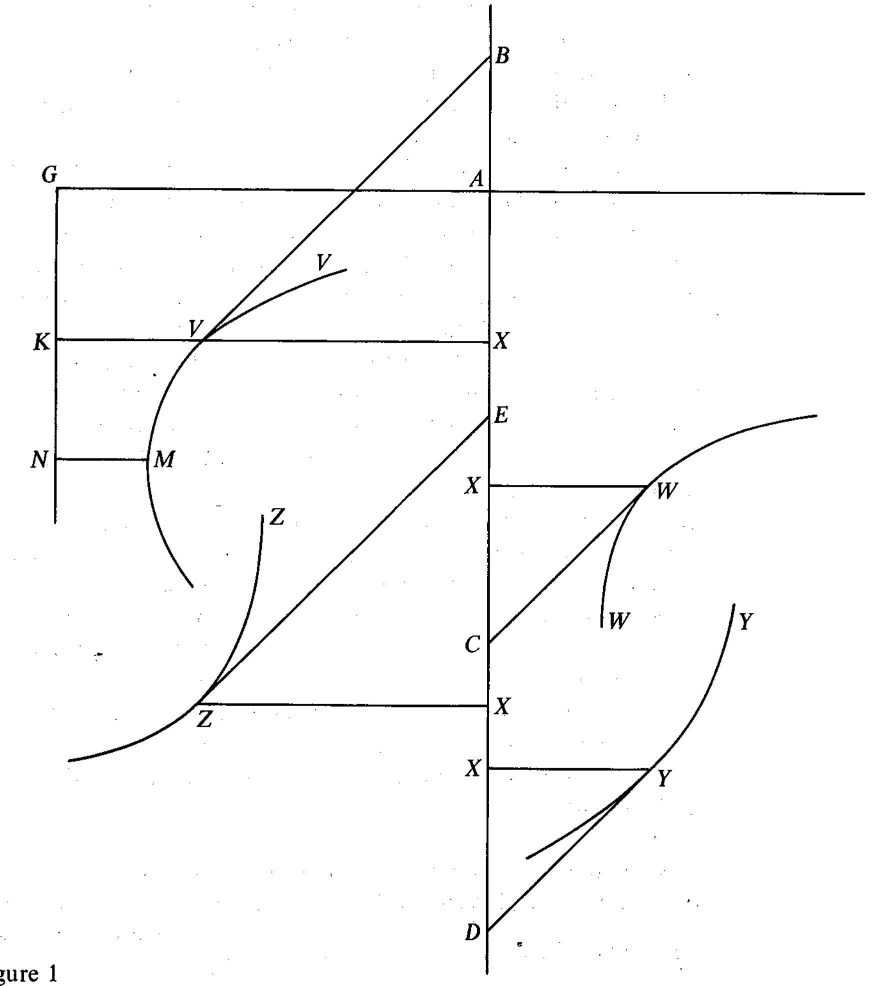

Figure 1

If a is a given constant, then $da = 0$, and $d(ax) = a\,dx$. If $y = v$ (that is, if the ordinate of any curve YY is equal to any corresponding ordinate of the curve VV), then $dy = dv$. Now *addition* and *subtraction*: if $z - y + w + z = v$, then $d(z - y + w + x) = dv = dz - dy + dw + dx$. Multiplication: $d(xv) = x\,dv + v\,dx$, or, setting $y = xv$, $dy = x\,dv + v\,dx$. It is indifferent whether we take a formula such as xv or its replacing letter such as y. It is to be noted that x and dx are treated in this calculus in the same way as y and dy, or any other indeterminate letter with its difference. It is also to be noted that we cannot always move backward from a differential equation without some caution, something which we shall discuss elsewhere.

Now *division*: $d\frac{v}{y}$ or $\left(\text{if } z = \frac{v}{y}\right) dz = \frac{\pm v\,dy \mp y\,dv}{yy}$.

The following should be kept well in mind about the *signs*. When in the calculus for a letter simply its differential is substituted, then the signs are preserved; for z we write dz, for $-z$ we write $-dz$, as appears from the previously given rule for addition and subtraction. However, when it comes to an explanation of the values, that is, when the relation of z to x is considered, then we can decide whether dz is a positive quantity or less than zero (or negative). When the latter occurs, then the tangent ZE is not directed toward A, but in the opposite direction, down from X. This happens when the ordinates z decrease with increasing x. And since the ordinates v sometimes increase and sometimes decrease, dv will sometimes be positive and sometimes be negative; in the first case the tangent VB is directed toward A, in the latter it is directed in the opposite sense. None of these cases happens in the intermediate position at M, at the moment when v neither increases nor decreases, but is stationary. Then $dv = 0$, and it does not matter whether the quantity is positive or negative, since $+0 = -0$. At this place v, that is, the ordinate LM, is *maximum* (or, when the convexity is turned to the axis, *minimum*), and the tangent to the curve at M is directed neither in the direction from X up to A, to approach the axis, nor down to the other side, but is parallel to the axis. When dv is infinite with respect to dx, then the tangent is perpendicular to the axis, that is, it is the ordinate itself. When $dv = dx$, then the tangent makes half a right angle with the axis. When with increasing ordinates v its increments or differences dv also increase (that is, when dv is positive, $d\,dv$, the difference of the differences, is also positive, and when dv is negative, $d\,dv$ is also negative), then the curve turns toward the axis its *concavity*, in the other case its *convexity*. Where the increment is maximum or minimum, or where the increments from decreasing turn into increasing, or the opposite, there is a *point of inflection*. Here concavity and convexity are interchanged, provided the ordinates too do not turn from increasing into decreasing or the opposite, because then the concavity or convexity would remain. However, it is impossible that the increments continue to increase or decrease, but the ordinates turn from increasing into decreasing, or the opposite. Hence a point of inflection occurs when $d\,dv = 0$ while neither v nor $dv = 0$. The problem of finding inflection therefore has not, like that of finding a maximum, two equal roots, but three. This all depends on the correct use of the signs.

Sometimes it is better to use *ambiguous signs*, as we have done with the division, before it is determined what the precise sign is. When with increasing x v/y increases (or decreases), then the ambiguous signs in $d\dfrac{v}{y} = \dfrac{\pm v\,dy \mp y\,dv}{yy}$ must be determined in such a way that this fraction is a positive (or negative) quantity. But $\mp$ means the opposite of $\pm$, so that when one is $+$ the other is $-$ or vice versa. There also may be several ambiguities in the same computation, which I distinguish by parentheses. For example, let $\dfrac{v}{y} + \dfrac{y}{z} + \dfrac{x}{v} = w$; then we must write

$$\frac{\pm v\,dy \mp y\,dv}{yy} + \frac{(\pm)y\,dz\,(\mp)\,z\,dy}{zz} + \frac{((\pm))x\,dv\,((\mp))\,v\,dx}{vv} = d[w],$$

so that the ambiguities in the different terms may not be confused. We must take notice that an ambiguous sign with itself gives $+$, with its opposite gives $-$, while with another ambiguous sign it forms a new ambiguity depending on both.

Powers. $dx^a = ax^{a-1}\,dx$; for example, $dx^3 = 3x^2dx$. $d\dfrac{1}{x^a} = -\dfrac{a\,dx}{x^{a[+]1}}$; for example, if $w = \dfrac{1}{x^3}$, then $dw = -\dfrac{3\,dx}{x^4}$.

Roots. $d\sqrt[b]{x^a} = \dfrac{a}{b}dx\sqrt[b]{x^{a-b}}$ (hence $d\sqrt[2]{y} = \dfrac{dy}{2\sqrt[2]{y}}$, for in this case $a = 1$, $b = 2$), therefore $\dfrac{a}{b}\sqrt[b]{x^{a-b}} = \frac{1}{2}\sqrt[2]{y^{-1}}$, but y^{-1} is the same as $\dfrac{1}{y}$; from the nature of the exponents in a geometric progression, and $\sqrt[2]{\dfrac{1}{y}} = \dfrac{1}{\sqrt[2]{y}}$, $d\dfrac{1}{\sqrt[b]{x^a}} = \dfrac{-a\,dx}{b\sqrt[b]{x^{a+b}}}$. The law for integral powers would have been sufficient to cover the case of fractions as well as roots, for a power becomes a fraction when the exponent is negative, and changes into a root when the exponent is fractional. However, I prefer to draw these conclusions myself rather than relegate their deduction to others, since they are quite general and occur often. In a matter that is already complicated in itself it is preferable to facilitate the operations.

Knowing thus the *Algorithm* (as I may say) of this calculus, which I call *differential calculus*, all other differential equations can be solved by a common method. We can find maxima and minima as well as tangents without the necessity of removing fractions, irrationals, and other restrictions, as had to be done according to the methods that have been published hitherto. The demonstration of all this will be easy to one who is experienced in these matters and who considers the fact, until now not sufficiently explored, that dx, dy, dv, dw, dz can be taken proportional to the momentary differences, that is, increments or decrements, of the corresponding x, y, v, w, z. To any given equation we can thus write its differential equation. This can be done by simply substituting for each *term* (that is, any part which through addition or subtraction contributes to the equation) its differential quantity. For any other quantity (not itself a term, but contributing to the formation of the term) we use its differential quantity, to form the differential quantity of the term itself, not by simple substitution, but according to the prescribed Algorithm. The methods published before have no such transition. They mostly use a line such as DX or of similar kind, but not the line dy which is the fourth proportional to DX, DY, dx—something quite confusing. From there they go on removing fractions and irrationals (in which undetermined quantities occur). It is clear that our method also covers transcendental curves—those that cannot be reduced by algebraic computation, or have no particular degree—and thus holds in a most general way without any particular and not always satisfied assumptions.

We have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the *curve*. This infinitely small distance can always be expressed by a known differential like dv, or by a relation to it, that is, by some known tangent. In particular, if y were a transcendental quantity, for instance the ordinate of a cycloid, and it entered into a computation in which z, the ordinate of another curve, were determined, and if we desired to know dz or by means of dz the tangent of this latter curve, then we should by all means determine dz by means of dy, since we have the tangent of the cycloid. The tangent to the cycloid itself, if we assume that we do not yet have it, could be found in a similar way from the given property of the tangent to the circle.

Now I shall propose an example of the calculus, in which I shall indicate division by $x{:}y$, which means the same as x divided by y, or $\dfrac{x}{y}$. Let the *first* or given equation be

$x:y + (a + bx)(c - xx):(ex + fxx)^2 + ax\sqrt{gg + yy} + yy:\sqrt{hh + lx + mxx} = 0$. It expresses the relation between x and y or between AX and XY, where a, b, c, e, f, g, h are given. We wish to draw from a point Y the line YD tangent to the curve, or to find the ratio of the line DX to the given line XY. We shall write for short $n = a + bx$, $p = c - xx$, $q = ex + fxx$, $r = gg + yy$, and $s = hh + lx + mxx$. We obtain $x:y + np:qq + ax\sqrt{r} + yy:\sqrt{s} = 0$, which we call the *second* equation. From our calculus it follows that

$$d(x:y) = (\pm x\,dy \mp y\,dx):yy,$$

and equally that

$$d(np:qq) = [(\pm)2np\,dq(\mp)q(n\,dp + p\,dn)]:q^3,$$

$$d(ax\sqrt{r}) = +ax\,dr:2\sqrt{r} + a\,dx\sqrt{r},$$

$$d(yy:\sqrt{s}) = ((\pm))yy\,ds\,((\mp))\,4ys\,dy:2s\sqrt{s}.$$

All these differential quantities from $d(x:y)$ to $d(yy:\sqrt{s})$ added together give 0, and thus produce a *third* equation, obtained from the terms of the second equation by substituting their differential quantities. Now $dn = b\,dx$ and $dp = -2x\,dx$, $d = e\,dx + 2fx\,dx$, $dr = 2y\,dy$, and $ds = l\,dx + 2mx\,dx$. When we substitute these values into the third equation we obtain a *fourth* equation, in which the only remaining differential quantities, namely dx, dy, are all outside of the denominators and without restrictions. Each term is multiplied either by dx or by dy, so that the law of homogeneity always holds with respect to these two quantities, however complicated the computation may be. From this we can always obtain the value of $dx:dy$, the ratio of dx to dy, or the ratio of the required DX to the given XY. In our case this ratio will be (if the fourth equation is changed into a proportionality):

$$\mp x:yy - axy:\sqrt{r}\,(\mp)\,2y:\sqrt{s}$$

divided by

$$\mp 1:y(\pm)(2npe + 2fx):q^3(\mp)(-2nx + pb):qq + a\sqrt{r}((\pm))yy(l + 2mx):2s\sqrt{s}.$$

Now x and y are given since point Y is given. Also given are the values of n, p, q, r, s expressed in x and y, which we wrote down above. Hence we have obtained what we required. Although this example is rather complicated we have presented it to show how the above-mentioned rules can be used even in a more difficult computation. Now it remains to show thair use in cases easier to grasp.

Let two points C and E [Figure 2] be given and a line SS in the same plane. It is required to find a point F on SS such that when E and C are connected with F the sum of the rectangle of CF and a given line h and the rectangle of FE and a given line r are as small as possible. In other words, if SS is a line separating two media, and h represents the density of the medium on the side of C (say water), r that of the medium on the side of E (say air), then we ask for the point F such that the path from C to E via F is the shortest possible. Let us assume that all such possible sums of rectangles, or all possible paths, are represented by the ordinates KV of curve VV perpendicular to the line GK [Figure 1]. We shall call these ordinates w. Then it is required to find their minimum NM. Since C and E [Figure 2] are given, their perpendiculars to SS are also given, namely CP (which we call c) and EQ (which we call e); moreover PQ (which we call p) is

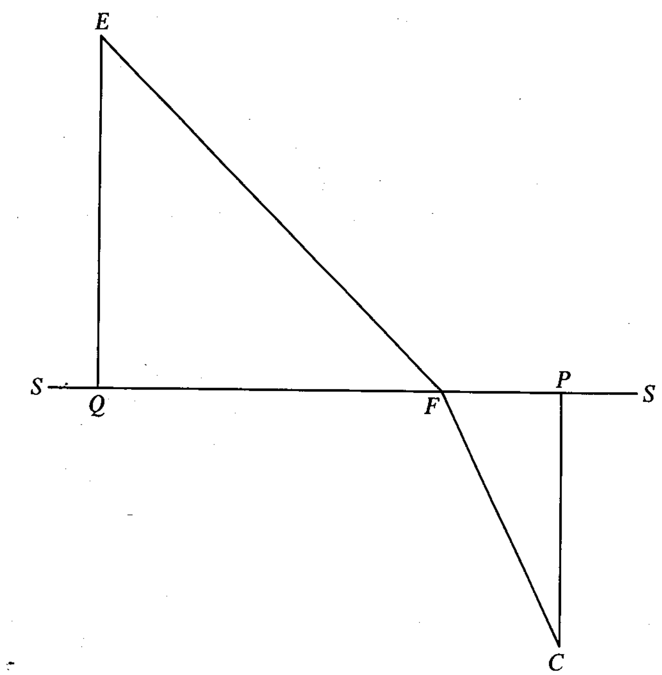

Figure 2

given. We denote $QF = GN$ (or AX) by x, CF by f, and EF by g. Then $FP = p - x$, $f = \sqrt{cc + pp - 2px + xx}$ or $= \sqrt{l}$ for short; $g = \sqrt{ee + xx}$ or $= \sqrt{m}$ for short. Hence

$$w = h\sqrt{l} + r\sqrt{m}.$$

The differential equation (since $dw = 0$ in the case of a minimum) is, according to our calculus,

$$0 = +h\,dl:2\sqrt{l} + r\,dm:2\sqrt{m}.$$

But $dl = -2(p' - x)\,dx$, $dm = 2x\,dx$; hence

$$h(p - x):f = rx:g.$$

When we now apply this to dioptrics, and take f and g, that is, CF and EF, equal to each other (since the refraction at the point F is the same no matter how long the line CF may be), then $h(p - x) = rx$ or $h:r = x:(p - x)$, or $h:r = QF:FP$; hence the sines of the angles of incidence and of refraction, FP and QF, are in inverse ratio to r and h, the densities of the media in which the incidence and the refraction take place. However, this density is not to be understood with respect to us, but to the resistance which the light rays meet. Thus we have a demonstration of the computation exhibited elsewhere in these *Acta*, where we presented a general foundation of optics, catoptrics, and dioptrics. Other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic.

This I shall explain by still another example. Let *13* [Figure 3] be a curve of such a nature that, if we draw from one of its points, such as *3*, six lines *34, 35, 36, 37, 38, 39* to six fixed points *4, 5, 6, 7, 8, 9* on the axis, then their sum is equal to a given line. Let