Numbers	Denomination
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1 024	10
2 048	11
4 096	12
8 192	13
16 384	14
32 768	15
65 536	16
131 072	17
262 144	18
524 288	19
1 048 576	20

Now it is necessary to know that 1 represents and is in the place of numbers, whose denomination is 0. 2 represents [...] the first terms, whose denomination is 1. 4 holds the place of the second terms, whose denomination is 2. And 8 is in the place of the third terms, 16 holds the place of the fourth terms, 32 represents the fifth terms, and so for the others. Now whoever multiplies 1 by 1, it comes to 1, and because 1 multiplied by 1 does not change at all, neither does any other number when it is multiplied by 1 increase or diminish, and for this consideration, whoever multiplies a number by a number, it comes to a number, whose denomination is 0. And whoever adds 0 to 0 makes 0. Afterwards, whoever multiplies 2, which is the first number, by 1, which is a number, the multiplication comes to 2; then afterwards, whoever adds their denominations, which are 0 and 1, it makes 1; thus the multiplication comes to 2^1. And from this it comes that when one multiplies numbers by first terms or vice versa, it comes to first terms. Also whoever multiplies 2^1 by 2^1, it comes to 4 which is a second number. Thus the multiplication amounts to 4^2. For 2 multiplied by 2 makes 4 and adding the denominations, that is, 1 with 1, makes 2. And from this it comes that whoever multiplies first terms by first terms, it comes to second terms. Likewise whoever multiplies 2^1 by 4^2, it comes to 8^3. For 2 multiplied by 4 and 1 added with 2 makes 8^3. And thus whoever multiplies first terms by second terms, it comes to third terms. Also, whoever multiplies 4^2 by 4^2, it comes to 16 which is a fourth number, and for this reason whoever multiplies second terms by second terms, it comes to fourth terms. Likewise whoever multiplies 4 which is a second number by 8 which is a third number makes 32 which is a fifth number. And thus whoever multiplies second terms by third terms or vice versa, it comes to fifth terms. And third terms by fourth terms comes to 7th terms, and fourth terms by fourth terms, it comes to 8th terms, and so for the others. In this discussion there is manifest a secret which is in the proportional numbers. It is that whoever multiplies a proportional number by itself, it comes to the number of the double of its denomination, as, whoever multiplies 8 which is a third number by itself, it comes to 64 which is a sixth. And 16 which is a fourth number multiplied by itself should come to 256, which is an eighth. And whoever multiplies 128 which is the 7th proportional by 512 which is the 9th, it should come to 65 536 which is the 16th.

7.B3 Luca Pacioli

(a) *On the content of his* Summa

The whole of this book is divided into five principal parts. In the first, numbers are discussed, in every way you would expect in simple and speculative practice. That is, writing and reading the characters, division, multiplication, addition, subtraction, and all sorts of progressions with very worthy rules newly induced, and very subtle cases; and the extraction of roots with numbers and with instruments and by geometrical methods, with their approximations. The philosophical algorisms discuss these things, from which, through this, there will always be knowledge of whole numbers, fractions, roots, binomials, their conjugates, and roots of roots, and every method of solving every proposed problem by algebra. And proportions and proportionalities and division, multiplication, addition, subtraction, which are necessary to perspective, music, astrology, cosmography, architecture, law, and medicine. With every substantiation from the fifth book of Euclid. And the rules of false position with their explanations. And of irrational lines, with which the whole of the tenth book of Euclid deals, with their practical methods of operation with clear demonstrations, always worked in such a way that everyone can learn them with great ease. And all these things with what follows will be according to the ancient and also modern mathematicians. Mostly from the very perspicacious philosopher of Megara, Euclid, and Severinus Boethius, and from our modern mathematicians, Leonardo Pisano [Fibonacci], Giordano, Biagio of Parma, Johannes de Sacrobosco, and Prodocimo of Padua, from whom I take the major part of this volume. You will have the table of this part and all the others, one by one below, set down in order, according to their distinctions, treatises, articles, and pages, according to how they differ, part by part. And in this first part are also contained all commercial occurrences of problems and rules, that is by hundredweights, thousands, pounds, ounces, investments, sales, profits, losses, journeys or transportations of goods, weights, measures, and money from place to place. And calculation of prices, with limitations of profit, loss, tares, gifts, uses, import and export duties in different places, taxes on sales made through brokers, carriage, fares, stabling, and whatever other exactions there may be, such as hiring, rents, household salaries, agents' fees, and workmen's wages. Appreciation, depreciation, gold, silver, copper, lightness and heaviness of all weights, superfluity and scarcity of all measures; lengths, widths, heights, and thicknesses, according to the commercial custom.

(b) *On algebra*

I do not think that I need now to defer any longer the greatest part necessary to the practice of arithmetic and also of geometry, called commonly 'the greater art', or 'the rule of the thing', or 'algebra and almucabala', called by us 'speculative practice'. Because higher things are contained in it than in the lesser art or business practice, as will be shown as we proceed. Such as roots and their squares, both simple and compound ways that occur; and in binomials and their conjugates; and in plus and minus abstractly upheld in its operation. These things cannot happen ordinarily in questions or problems of trade, unless so many new impositions were made, raising the subject to a higher level of operation. Wanting to deal with this subject in such a way as to proceed in an orderly manner, we will divide this section into seven principal parts. In the first of these we will speak of the two terms found to be convenient to this kind of operation, one called 'plus' and the other 'minus', and how between them they are used in their workings, and why they were invented. They are the most necessary of all to this practice. Then in the second we will deal with roots in every way that they are used and worked, with their definitions and divisions or disparities. In the third we will tell of binomials and also conjugates or residues, and of their disparities and operations. And of the notion of the fifteen lines, with which all the tenth book of Euclid principally deals. In the fourth we shall show certain ways in multiplication, and consequently in division, of ordering the *cosa*, and squares and cubes and higher powers, set down in the way of common tables, so that in the multiplication of these powers it can be found more easily what one of these quantities times another will generate. And the same with dividing one by another, what should come of such a division, which will be obvious from the information on multiplication. In the fifth will be set down the basic equations of algebra and almucabala with their distinctions between simple and compound, and together with these, the proof of each of them, and where their strength comes from, with clear and open diagrams. In the sixth we shall demonstrate the method of constructing the basic equation appropriate to whatever equation the performer may come across, so that he can with great ease make reply to such an equation, which will conform to the solution of that basic equation, for there are six fundamental and principal forms (as will be described) and infinite then are those which have their origin in these. And they proceed according to the proportion of these six forms. In the seventh and last principal part we shall explain why it is not possible to solve every question by algebra, and in consequence we shall give information of each impossibility, according to the way of its equation. And thus, by looking at any equation one will know its possibility. And after all these things, several very useful questions will be set.

(c) *On quadratic equations*

We have seen what these terms mean and represent in the practice of arithmetic. Now we must see in how many ways they can be made equal, one to the other, and the other to the one, and two of them to one of them, and one to two of them. On this I say that they can be made equal to each other in six ways. First, the square to the things. Second, the square to the numbers. Third, thing or things to numbers. Fourth, the square and the thing can be put equal to the number. Fifth, the thing and the number can be equal to the square. Sixth, the square and the number can be equal to the thing. Other than in these six ways described it is not possible to have any equation in them. And in regard to these six equations, there have been formed six rules, which are commonly called the six basic equations. And of these six, three give a standard to the first three of the equations, that is, of one to the other and the other to the one, which are called simple. And the other three are standard forms of the other three equations, that is, of two equal to one and conversely. And these three are called compound, that is, that two together are always equal to the third, which happens (as we have said) in three ways, whence the rules or cases of one and the other. These are the standard three of the simple equations. [...]

The three compound rules of algebraic equations

1 When the squares and the things are equal to a number, first you must reduce all the equation to one square, that is if there is less than one square you must equally restore and make good. And if there is more than one square you must reduce to one square, and reducing is done by dividing the whole of the equation by the amount of the squares. And when you have done this, halve the things, and multiply one half by itself. The number is added to this product, and the root of this sum minus the half of the things is the value of the thing required.

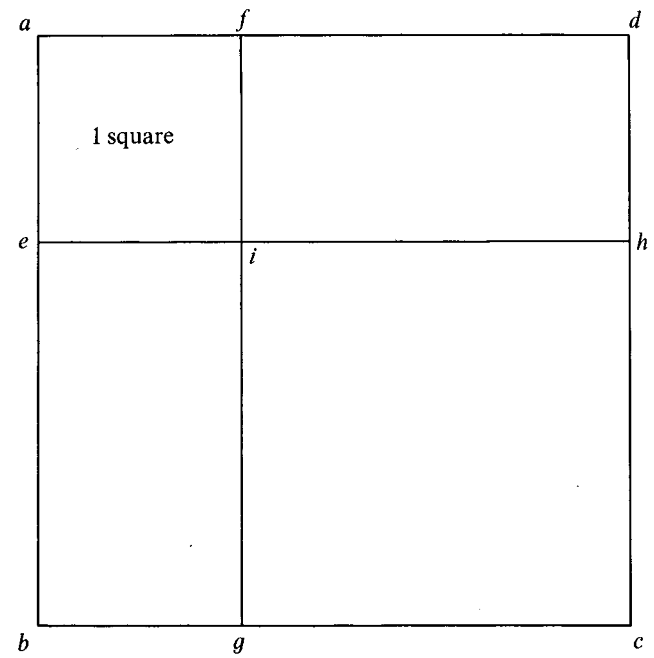
2 When the thing and the number are equal to squares. First (the same thing is done as above in the preceding, that is) reduce the whole equation to one square, removing anything greater than one square equally and geometrically, and making up anything less on both sides likewise geometrically, by doing this: (as I said above) it suffices to divide the whole of the equation by the quantity of squares, and then it will be reduced to one square. Having done this you will halve the things, and multiply one half by itself, and to the result add the number. And the root of this sum will always be the value of the thing, when the half of the things is added to it.

3 When the square and the number are equal to the things. In that case (do as we said above, that is) reduce the whole equation to one square, that is, divide the whole of it by the quantity of the squares, and then halve the things and multiply one half by itself. And from the result always subtract the number which is found in the equation, and the root of the remainder added to the half of the things, or indeed subtracted from the half of the things, will be the value of the thing.

Geometrical demonstration of the first compound basic equation

[...] the other three compound equations certainly need a cautious declaration and demonstration, so that their truthfulness will be matched more openly, that is that one must observe in their setting up what is contained in their statements above. And here in the following I intend to demonstrate them one by one in an orderly way. And first we shall demonstrate the truth of the one where the squares and the things are equal to a number. For example, 1 square and 10 things are equal to 39, which are straightforward numbers. [...] Let *abcd* be a tetragon which has each side greater than the number 5. And on the side *ab* is marked the point *e* in such a way that *be* is exactly the number 5 and the remainder *ea* is an unknown quantity. And in the same way, on the side *ad* is marked the point *f* in such a way that *fd* is again the number 5, and *af* is again an unknown excess over 5. And in the same way, on the side *bc* is marked the point *g*, so that again *cg* is 5, and the remainder *bg* is similarly unknown. And on the side cd, the point h in the same way, that is so that ch is 5, and hd unknown. Thus of all these 4 straight lines, each will be known to be 5.

[Pacioli goes on to deduce very carefully that the side ab is 8, and so the root, ea, is 3.]



8 Sixteenth-century European Mathematics

8.A The Development of Algebra in Italy

The first person to solve cubic equations algebraically was Scipione del Ferro (1465?–1525), who was professor of mathematics at Bologna. At some stage he entrusted the solution method to his pupil Antonio Maria Fior, who proceeded to live off it by challenging others to contests at mathematical problem-solving. As can be seen from 8.A1, Fior thought it worth while to put all his mathematical eggs into this cubical basket, which rather suggests that he was not a very good mathematician but very confident of his secret. Indeed, because all his problems are of the form 'cube and things equal to numbers' ($x^3 + px = q$), it seems very likely that this is the only kind of cubic del Ferro had taught him to solve. But in 1535 he was unlucky enough to challenge Niccolò Tartaglia (1506?–1559) who, on the night of 12–13 February, worked out the solution for himself, and so won the contest. Apparently he declined the thirty dinners at Fior's expense he had thereby won. The news of his success soon spread, and Gerolamo Cardano (1501–1576) heard of it. After much patient lobbying, they met in 1539 in Milan, and Tartaglia divulged the method. Tartaglia's later claims as to what was said on this occasion form 8.A2. (The use of verses to remember complicated items was not unusual at the time, and was proof against theft.) But when Cardano and his pupil Ludovico Ferrari (1522–1565) learned that the solution method had been known to del Ferro, and found that they had new things to say, both about other types of cubic and about quartic equations (which Ferrari had discovered how to solve), they decided to publish (see 8.A4). Tartaglia was furious, and a prolonged battle was waged between them, some of which can be glimpsed in 8.A3. Some of Ferrari's problems, in particular, show that he had a broad and philosophically rich attitude to mathematics.

Even by Rafael Bombelli's time, a mere generation later, things had begun to change. Bombelli's attitude to algebra was heavily influenced by the rediscovery of ancient texts then well under way (see Section 8.B) and,