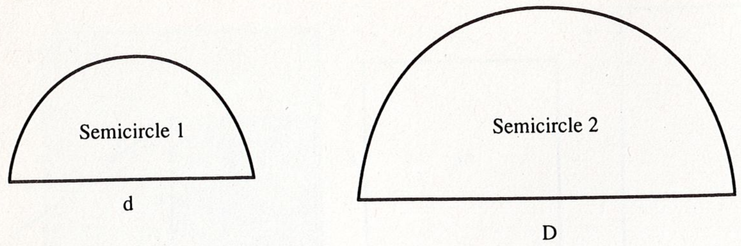

FIGURE 1.14

The first two of these results were well known long before Hippocrates came upon the scene. The last proposition, on the other hand, is considerably more sophisticated. It gives a comparison of the areas of two circles or semicircles based on the relative areas of the squares constructed on their diameters (see Figure 1.14). For instance, if one semicircle has five times the diameter of another, the former has 25 times the area of the latter. This proposition presents math historians with a problem, for there is widespread doubt that Hippocrates actually had a valid proof. He may well have *thought* he could prove it, but modern scholars generally feel that this theorem—which later appeared as the second proposition in Book XII of Euclid's *Elements*—presented logical difficulties far beyond what Hippocrates would have been able to handle. (A derivation of this result is presented in Chapter 4.)

That aside, we now consider Hippocrates' proof. Begin with a semicircle having center O and radius $\overline{AO} = \overline{OB}$, as shown in Figure 1.15. Construct OC perpendicular to AB, with point C on the semicircle, and draw lines AC and BC. Bisect AC at D, and using $\overline{AD}$ as a radius and D as center, draw semicircle AEC, thus creating lune $AECF$, which is shaded in the diagram.

Hippocrates' plan of attack was simple yet brilliant. He first had to establish that the lune in question had *precisely* the same area as the shaded $\triangle AOC$. With this behind him, he could then apply the known fact that triangles can be squared to conclude that the lune can be squared as well. The details of the classic argument follow:

THEOREM Lune $AECF$ is quadrable.

PROOF Note that $\angle ACB$ is right since it is inscribed in a semicircle. Triangles AOC and BOC are congruent by the "side-angle-side" congruence scheme, and consequently $\overline{AC} = \overline{BC}$. We thus apply the Pythagorean theorem to get

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 = 2(\overline{AC})^2$$

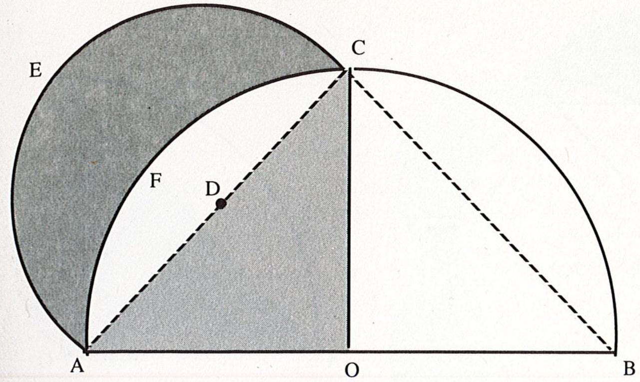

FIGURE 1.15

Because AB is the diameter of semicircle ACB, and AC is the diameter of semicircle AEC, we can apply the third principle above to get

$$\frac{\text{Area (semicircle } AEC)}{\text{Area (semicircle } ACB)} = \frac{(\overline{AC})^2}{(\overline{AB})^2} = \frac{(\overline{AC})^2}{2(\overline{AC})^2} = \frac{1}{2}$$

In other words, semicircle AEC has half the area of semicircle ACB.

But we now look at quadrant $AFCO$ (a "quadrant" is a quarter of a circle). Clearly this quadrant also has half the area of semicircle ACB, and we immediately conclude that

$$\text{Area (semicircle } AEC) = \text{Area (quadrant } AFCO)$$

Finally, we need only subtract from each of these figures their shared region $AFCD$, as in Figure 1.16. This leaves

$$\begin{aligned}\text{Area (semicircle } AEC) &- \text{Area (region } AFCD)\\ &= \text{Area (quadrant } AFCO) - \text{Area (region } AFCD)\end{aligned}$$

and a quick look at the diagram verifies that this amounts to

$$\text{Area (lune } AECF) = \text{Area } (\triangle ACO)$$

But, as we have seen, we can construct a square whose area equals that of the triangle, and thus equals that of the lune as well. This is the quadrature we sought.

Q.E.D.