First table.
10000000.0000000
1.0000000
9999999.0000000
.9999999
9999998.0000001
.9999998
9999997.0000003
to be continued up to
9999900.0004950

Thus from radius, with seven cyphers added for greater accuracy, namely, 10000000.0000000, subtract 1.0000000 you get 9999999.0000000; from this subtract .9999999, you get 9999998.0000001; and proceed in this way...until you create a hundred proportionals, the last of which, if you have computed rightly, will be 9999900.0004950.

17. The Second table proceeds from radius with six cyphers added, through fifty other numbers decreasing proportionally in the proportion which is easiest, and as near as possible to that subsisting between the first and last numbers of the First table.

Second table.
10000000.000000
100.000000
9999900.000000
99.999000
9999800.001000
to be continued up to
9995001.222927

Thus the first and last numbers of the First table are 10000000.0000000 and 9999900.0004950, in which proportion it is difficult to form fifty proportional numbers. A near and at the same time an easy proposition is 100000 to 99999, which may be continued with sufficient exactness by adding six cyphers to radius and continually subtracting from each number its own 100000th part... and this table contains, besides radius which is the first, fifty other proportional numbers, the last of which, if you have not erred, you will find to be 9995001.222927.[4]

Article 18 has a Third table of 69 columns, from 10^{12} down by 2000th parts to 9900473.57808.

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19. The first numbers of all the columns must proceed from radius with four cyphers added, in the proportion easiest and nearest to that subsisting between the first and the last numbers of the first column.

As the first and the last numbers of the first column are 10000000.0000 and 9900473.5780, the easiest proportion very near to this is 100 to 99. Accordingly sixty-eight numbers are to be continued from radius in the ratio of 100 to 99 by subtracting from each one of them its hundredth part.

20. In the same proportion a progression is to be made from the second number of the first column through the second numbers in all the columns, and from the third through the third, and from the fourth through the fourth, and from the others respectively through the others.

[4] This should be 9995001.224804.

Thus from any number in one column, by subtracting its hundredth part, the number of the same rank in the following column is made, and the numbers should be placed in order as follows.

Here follows a table of "Proportionals of the Third Table," with 69 columns, the last number in the sixty-ninth column being 4998609.4034, roughly half the original number 10000000.0000.

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21. Thus, in the Third table, between radius and half radius, you have sixty-eight numbers interpolated, in the proportion of 100 to 99, and between each two of these you have twenty numbers interpolated in the proportion of 10000 to 9995; and again, in the Second table, between the first two of these namely between 10000000 and 9995000, you have fifty numbers interpolated in the proportion of 100000 to 99999; and finally, in the First table, between the latter, you have a hundred numbers interpolated in the proportion of radius or 10000000 to 9999999; and since the difference of these is never more than unity, there is no need to divide it more minutely by interpolating means, whence these three tables, after they have been completed, will suffice for computing a Logarithmic table.

Hitherto we have explained how we may most easily place in tables sines or natural numbers progressing in geometrical proportion.

22. It remains, in the Third table at least, to place beside the sines or natural numbers decreasing geometrically their logarithms or artificial numbers increasing arithmetically.

Articles 23 and 24 represent arithmetic increase and geometric decrease by points on a line.

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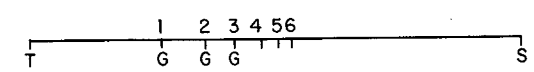
25. Whence a geometrically moving point approaching a fixed one has its velocities proportionate to its distances from the fixed one.

Thus referring to the preceding figure [Fig. 1], I say that when the geometrically moving point G is at T, its velocity is as the distance TS, and when

Fig. 1


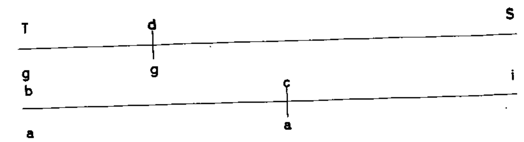
G is at 1 its velocity is as $1S$, and when at 2 its velocity is as $2S$, and so of the others. Hence, whatever be the proportion of the distances TS, $1S$, $2S$, $3S$, $4S$, etc., to each other, that of the velocities of G at the points T, 1, 2, 3, 4, etc., to one another, will be the same.

For we observe that a moving point is declared more or less swift, according as it is seen to be borne over a greater or less space in equal times. Hence the ratio of the spaces traversed is necessarily the same as that of the velocities. But the ratio of the spaces traversed in equal times, $T1$, 12, 23, 34, 45, etc., is that of the distances TS, $1S$, $2S$, $3S$, $4S$, etc. Hence it follows that the ratio to one another of the distances of G from S, namely TS, $1S$, $2S$, $3S$, $4S$, etc., is the same as that of the velocities of G at the points T, 1, 2, 3, 4, etc., respectively.

26. The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine.

Let the line TS [Fig. 2] be the radius, and dS a given sine in the same line; let g move geometrically from T to d in certain determinate moments of time. Again, let bi be another line, infinite towards i, along which, from b, let a move arithmetically with the same velocity as g had at first when at T; and from the fixed point b in the direction of i let a advance in just the same moments of time up to the point c. The number measuring the line bc is called the logarithm of the given sine dS.[5]

Fig. 2



27. Whence nothing is the logarithm of radius [*Unde sinus totius nihil est pro artificiali*]...

28. Whence also it follows that the logarithm of any given sine is greater than the difference between radius and the given sine, and less than the difference between radius and the quantity which exceeds it in the ratio of radius to the given sine. And these differences are therefore called the limits of the logarithm.

Thus, the preceding figure being repeated [Fig. 3], and ST being produced beyond T to o, so that oS is to TS as TS to dS, I say that bc, the logarithm of the sine dS, is greater than Td and less than oT. For in the same time that g is borne from o to T, g is borne from T to d, because (by 24) oT is such a part of oS as Td is of TS, and in the same time (by the definition of a logarithm) is a borne from b to c; so that oT, Td, and bc are distances traversed in equal times. But since g when moving between T and o is swifter than at T, and

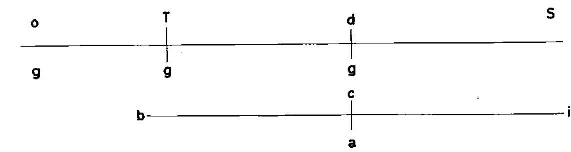
[5] In the language of the calculus: let $TS = a$ $(= 10^7)$, $dS = y$; then the initial velocity $(t = c)$ at g is a (see Art. 25), hence the velocity of g at d is $(d/dt)(a - y) = -dy/dt = y$, hence $y = ae^{-t}$. When $bc = x$, then $x = at =$ Nap log y. Hence Nap log $y = a \ln a/y$, so that (by Art. 27) for $y = a$, Nap log $a = 0$, where $\ln = \log_e$, the natural logarithm. The familiar rules for logarithmic computation do not apply:

$$\text{Nap log } xy = a(\ln a - \ln x - \ln y).$$

We should not be confused by the terms "radius" and "sine"; what is meant is a line segment TS and a section $dS \leqq TS$. When $a = 1$ the Nap log and the ln differ only in sign; this may have caused the confusion in some textbooks, which insist on calling the natural logarithms Napierian or Neperian logarithms.

between T and d slower, but at T is equally swift with a (by 26); it follows that oT the distance traversed by g moving swiftly is greater, and Td the distance traversed by g moving slowly is less, than bc the distance traversed by the point a with its medium motion, in just the same moments of time; the latter is, consequently, a certain mean between the two former. Therefore oT is called the greater limit, and Td the less limit of the logarithm which bc represents.

Fig. 3



29. Therefore to find the limits of the logarithm of a given sine.

By the preceding it is proved that the given sine being subtracted from radius the less limit remains, and that radius being multiplied into the less limit and the product divided by the given sine, the greater limit is produced, as in the following example.

30. Whence the first proportional of the First table, which is 9999999, has its logarithm between the limits 1.0000001 and 1.0000000...

31. The limits themselves differing insensibly, they or anything between them may be taken as the true logarithm...

32. There being any number of sines decreasing from radius in geometrical proportion, of one of which the logarithm or its limits is given, to find those of the others.

This necessarily follows from the definitions of arithmetical increase, of geometrical decrease, and of a logarithm... So that, if the first logarithm corresponding to the first sine after radius be given, the second logarithm will be double of it, the third triple, and so of the others; until the logarithms of all the sines be known...

33. Hence the logarithms of all the proportional sines of the First table may be included between near limits, and consequently given with sufficient exactness...

34. The difference of the logarithms of radius and a given sine is the logarithm of the given sine itself...

35. The difference of the logarithms of two sines must be added to the logarithm of the greater that you may have the logarithm of the less, and subtracted from the logarithm of the less that you may have the logarithm of the greater...

36. The logarithms of similarly proportioned sines are equidifferent.

This necessarily follows from the definitions of a logarithm and of the two motions... Also there is the same ratio of equality between the differences of the respective limits of the logarithms, namely as the differences of the less among themselves, so also of the greater among themselves, of which logarithms the sines are similarly proportioned.

37. Of three sines continued in geometrical proportion, as the square of the mean equals the product of the extremes, so of their logarithms the double of