12 Isaac Newton

Sir Isaac Newton (1642–1727) did not dominate even English mathematical and scientific life until the successful publication of his *Philosophiae Naturalis Principia Mathematica* (*The Mathematical Principles of Natural Philosophy*) in 1687 (see 12.B). Until then the vast bulk of his discoveries lay in his desk drawers, known only in outline to a few friends and colleagues. Afterwards, when he left Cambridge and moved to London, where he was made director of the Royal Mint, he began to publish increasingly, but even so it has remained for modern scholars to print more of Newton's mathematical work than Newton ever did. So the picture of Newton that we have, and the nature of his influence, are necessarily complicated. His first love was for mathematics, and his initial years at Cambridge were spent mastering the literature; works by Oughtred, Wallis, and especially Descartes's *Geometry* and the numerous commentaries on it. But soon he left them behind, and in 1664 began to do his own original research. Our first selections show him at work in this period investigating curves in the Cartesian style, but insisting on the centrality of the problem of tangents (see 12.A1, 12.A3, 12.A5). His use of infinite series lent his work a generality which surpassed Descartes's (see 12.A2, 12.A3), but two other features of his thought are also particularly noteworthy: his emphasis on the tangent as the instantaneous direction of motion along the curve; and his discovery of a pattern in the results which yielded him an algorithm (see 12.A4, 12.A5). Soon he realized that quadrature problems were inverse to tangency problems, and he was then in possession of what can be called the Newtonian calculus.

This calculus makes certain kinds of problems easy which had been difficult, and suggested to Newton that it was now possible to tackle a much harder problem, the inverse tangent problem (raised in 12.A6), which he regarded as a generalization of the finding of areas. His formal rules for 'differentiation' or finding fluxions did not, of course, invert in any simple way, but he found, as his two letters to Leibniz make clear (see 12.C), that his method of infinite series was a great help here too, although it was not the only method.

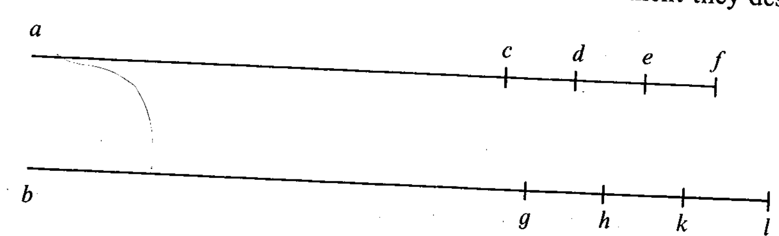
But Newton did not only invent the calculus, and write the *Principia* and his *Opticks*; he was also a first-rate geometer. By this we do not only mean that his geometrical arguments in the *Principia* are skilful and elegant, as they indeed are, but that there and elsewhere he had dramatic new things to say. Another fruit of the 1660s that was not to see the light until his *Opticks* was published in 1704 was his remarkable classification of cubic curves (see 12.D2), which was both a *tour de force* of Cartesian algebraic methods, and the occasion for a lapidary statement of the power of projective geometry. The work was to provoke many eighteenth-century geometers, some of whom we shall look at in 14.D. In the *Principia* itself he not only naturally included many results about conics satisfying various conditions, but also presented his projective transformations of curves (see 12.D4); this, too, gradually brought forth a literature of its own. However, as he grew older, he developed increasingly firm views on the subject of geometry and its superiority over algebra; 12.D3 is just one of many similar passages on this theme.

12.A Newton's Invention of the Calculus

12.A1 Tangents by motion and by the *o*-method

Lemma

If two bodys A, B, move uniformely the $\begin{matrix}\text{one}\\\text{other}\end{matrix}$ from $\begin{matrix}a\\b\end{matrix}$ to $\begin{matrix}c, d, e, f,\\g, h, k, l,\end{matrix}$ &c: in the same time. Then are the lines $\begin{matrix}ac,\\bg,\end{matrix}$ & $\begin{matrix}cd,\\gh,\end{matrix}$ & $\begin{matrix}de,\\hk,\end{matrix}$ & $\begin{matrix}ef,\\kl,\end{matrix}$ &c: as their velocitys $\frac{p}{q}$. And though they move not uniformely yet are the infinitely little lines which each moment they describe, as



their velocitys which they have while they describe them. As if the body A with the velocity p describe the infinitely little line ($cd =$) $p \times o$ in one moment, in that moment the body B with the velocity q will describe the line ($gh =$) $q \times o$. For $p{:}q{::}po{:}qo$. Soe that if the described lines bee ($ac =$) x, & ($bg =$) y, in one moment, they will bee ($ad =$) $x + po$, & ($bh =$) $y + qo$ in the next.

Demonstration

Now if the equation expressing the relation twixt the lines x & y bee $x^3 - abx + a^3 - dyy = 0$. I may substitute $x + po$ & $y + qo$ into the place of x & y; because (by the lemma) they as well as x & y, doe signify the lines described by the bodys A & B. By doeing so there results

$$x^3 + 3poxx + 3ppoox + p^3o^3 - dyy - 2dqoy - dqqoo = 0.$$
$$-abx \quad -abpo$$
$$+a^3$$

But $x^3 - abx + a^3 - dyy = 0$ (by supp). Therefore there remaines onely

$$3poxx + 3ppoox + p^3o^3 - 2dqoy - dqqoo = 0.$$
$$-abpo$$

Or dividing it by o tis

$$3px^2 + 3ppox + p^3oo - 2dqy - dqqoo = 0.$$
$$-abp$$

Also those termes are infinitely little in which o is. Therefore omitting them there rests

$$3pxx - abp - 2dqy = 0.$$

The like may bee done in all other equations.

12.A2 Rules for finding areas

The general method which I had devised some time ago for measuring the quantity of curves by an infinite series of terms you have, in the following, rather briefly explained than narrowly demonstrated.

To the base AB of some curve AD let the ordinate BD be perpendicular and let AB be called x and BD y. Let again $a, b, c, \ldots$ be given quantities and m, n integers. Then

Rule 1
If $ax^{m/n} = y$, then will $[na/(m+n)]x^{(m+n)/n}$ equal the area ABD. The matter will be evident by example.

Example 1 If $x^2(= 1 \times x^{\frac{2}{1}}) = y$, that is, if $a = n = 1$ and $m = 2$, then $\frac{1}{3}x^3 = ABD$.

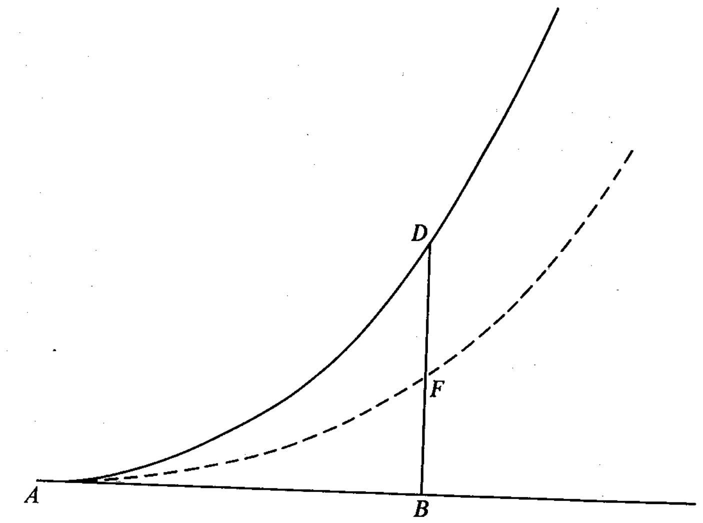
Example 2 If $4\sqrt{x}(= 4x^{\frac{1}{2}}) = y$, then $\frac{8}{3}x^{\frac{3}{2}}(= \frac{8}{3}\sqrt{x^3}) = ABD$.

Example 3 If $\sqrt[3]{x^5}(= x^{\frac{5}{3}}) = y$, then $\frac{3}{8}x^{\frac{8}{3}}(= \frac{3}{8}\sqrt[3]{x^8}) = ABD$.

Example 4 If $(1/x^2)(= x^{-2}) = y$, that is, if $a = n = 1$ and $m = -2$, then $([1/-1]x^{-\frac{1}{1}} =) - x^{-1}(= -[1/x]) = \alpha BD$ infinitely extended in the direction of α: the computation sets its sign negative because it lies on the further side of the line BD.

Example 5 If $2/3\sqrt{x^3}(= \frac{2}{3}x^{-\frac{3}{2}}) = y$, then $(2/-1)x^{-\frac{1}{2}} = -(2/\sqrt{x}) = BD\alpha$.

Example 6 If $(1/x)(= x^{-1}) = y$, then $(1/0)x^{\frac{0}{1}} = (1/0)x^0 = (1/0) \times 1 = \infty$, just as the area of the hyperbola is on each side of the line BD.



Rule 2
If the value of y is compounded of several terms of that kind the area also will be compounded of the areas which arise separately from each of those terms.

Let its first examples be these. If $x^2 + x^{\frac{3}{2}} = y$, then $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} = ABD$. For if there be always $BF = x^2$ and $FD = x^{\frac{3}{2}}$, then by the preceding rule $\frac{1}{3}x^3 =$ the surface AFB described by the line BF and $\frac{2}{5}x^{\frac{5}{2}} = AFD$ described by DF; and consequently $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ the whole ABD. Thus if $x^2 - x^{\frac{3}{2}} = y$, then $\frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}} = ABD$: and if $3x - 2x^2 + x^3 - 5x^4 = y$, then $\frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5 = ABD$.

12.A3 The sine series and the cycloid

If it is desired to find the sine AB from the arc αD given, of the equation $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 \ldots$ found above (supposing namely that $AB = x$, $\widehat{\alpha D} = z$ and $A\alpha = 1$) I extract the root, which will be $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 \ldots$. If, moreover, you want the cosine $A\beta$ of that given arc, make $A\beta(= \sqrt{[1 - x^2]}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10} \ldots$.

Let it be noted here, by the way, that when you know 5 or 6 terms of those roots you will for the most part be able to prolong them at will by observing analogies. Thus you may prolong this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 \ldots$ by dividing the last term by these numbers in order $2, 3, 4, 5, 6, 7, \ldots$; and this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 \ldots$ by these $2 \times 3, 4 \times 5, 6 \times 7, 8 \times 9, 10 \times 11, \ldots$; and this $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 \ldots$ by these 1×2, 3×4, 5×6, 7×8, $9 \times 10, \ldots$; while this $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 \ldots$ you may produce by multiplying by these $\frac{1 \times 1}{2 \times 3}, \frac{3 \times 3}{4 \times 5}, \frac{5 \times 5}{6 \times 7}, \frac{7 \times 7}{8 \times 9}, \ldots$. And so for others.

Let these remarks suffice for geometrical curves. But, indeed, if the curve is mechanical it yet by no means spurns our method. Take, for example, the cycloid

$ADFG$ whose vertex is A and axis AH while AKH is the 'wheel' by which it is generated. And let the surface ABD be sought. Setting now $AB = x$, $BD = y$ (as above) and $AH = 1$, I seek in the first instance the length of BD. Precisely, by the nature of a cycloid, KD is equal to the arc AK, and therefore the whole line $BD = BK +$ the arc AK. But $BK(=\sqrt{[x - x^2]}) = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} \ldots$ and (from the preceding) the arc $AK = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} \ldots$, so that in consequence the whole line $BD = 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} \ldots$. And (by Rule 2) the area $ABD = \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} \ldots$.

Or more briefly thus. Since the straight line AK is parallel to the tangent TD, AB will be to BK as the momentum of the line AB to the momentum of the line BD, that is, $x{:}\sqrt{[x - x^2]} = 1{:}x^{-1}\sqrt{[x - x^2]}$ or $x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}} \ldots$. Hence (by Rule 2) $BD = 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}} \ldots$ and the surface $ABD = \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}} \ldots$.

In a not dissimilar way you will (on setting C as the circle's centre and $CB = x$) obtain the area $CBDF$, and so on.

12.A4 Quadrature as the inverse of fluxions

Let any curve $AD\delta$ have base $AB = x$, perpendicular ordinate $BD = y$ and area $ABD = z$. Take $B\beta = o$, $BK = v$ and the rectangle $B\beta HK(ov)$ equal to the space $B\beta\delta D$. It is, therefore, $A\beta = x + o$ and $A\delta\beta = z + ov$. With these premisses, from any arbitrarily assumed relationship between x and z I seek y in the way you see following.

Take at will $\frac{2}{3}x^{\frac{3}{2}} = z$ or $\frac{4}{9}x^3 = z^2$. Then, when $x + o(A\beta)$ is substituted for x and $z + ov(A\delta\beta)$ for z, there arises (by the nature of the curve) $\frac{4}{9}(x^3 + 3x^2o + 3xo^2 + o^3) = z^2 + 2zov + o^2v^2$. On taking away equal quantities ($\frac{4}{9}x^3$ and z^2) and dividing the rest by o, there remains $\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2$. If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish and there will consequently remain $\frac{4}{9} \times 3x^2 = 2zv$ or $\frac{2}{3}x^2(= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, that is, $x^{\frac{1}{2}}(= x^2/x^{\frac{3}{2}}) = y$. Conversely therefore if $x^{\frac{1}{2}} = y$, then will $\frac{2}{3}x^{\frac{3}{2}} = z$.

Or in general if $[n/(m + n)]ax^{(m+n)/n} = z$, that is, by setting $na/(m + n) = c$ and $m + n = p$, if $cx^{p/n} = z$ or $c^nx^p = z^n$, then when $x + o$ is substituted for x and $z + ov$ (or, what is its equivalent, $z + oy$) for z there arises $c^n(x^p + pox^{p-1} \ldots) = z^n + noyz^{n-1} \ldots$, omitting the other terms, to be precise, which would ultimately vanish. Now, on taking away the equal terms c^nx^p and z^n and dividing the rest by o, there remains $c^npx^{p-1} = nyz^{n-1}(= nyz^n/z) = nyc^nx^p/cx^{p/n}$. That is, on dividing by c^nx^p, there will be $px^{-1} = ny/cx^{p/n}$ or $pcx^{(p-n)/n} = y$; in other words, by restoring $na/(m + n)$ for o and $m + n$ for p, that is, m for $p - n$ and na for pc, there will come $ax^{m/n} = y$. Conversely therefore if $ax^{m/n} = y$, then will $[n/(m + n)]ax^{(m+n)/n} = z$. As was to be proved.

Here in passing may be noted a method by which as many curves as you please whose areas are known may be found: namely, by assuming any equation at will for the relationship between the area z and from it in consequence seeking the ordinate y. So if you should suppose $\sqrt{[a^2 + x^2]} = z$, by computation you will find $x/\sqrt{[a^2 + x^2]} = y$. And similarly in other cases.

12.A5 Finding fluxions of fluent quantities

The moments of the fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed $\dot{x}$ and an infinitely small quantity o (that is, by $\dot{x}o$), then the moments of the others, v, y, z, [...], will be expressed by $\dot{v}o, \dot{y}o, \dot{z}o$, [...] seeing that $\dot{v}o, \dot{x}o, \dot{y}o$ and $\dot{z}o$ are to one another as $\dot{v}, \dot{x}, \dot{y}$ and $\dot{z}$.

Now, since the moments (say, $\dot{x}o$ and $\dot{y}o$) of fluent quantities (x and y say) are the infinitely small additions by which those quantities increase during each infinitely small interval of time, it follows that those quantities x and y after any infinitely small interval of time will become $x + \dot{x}o$ and $y + \dot{y}o$. Consequently, an equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between $x + \dot{x}o$ and $y + \dot{y}o$ as between x and y; and so $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in place of the latter quantities, x and y, in the said equation.

Let there be given, accordingly, any equation $x^3 - ax^2 + axy - y^3 = 0$ and substitute $x + \dot{x}o$ in place of x and $y + \dot{y}o$ in place of y: there will emerge

$$(x^3 + 3\dot{x}ox^2 + 3\dot{x}^2o^2x + \dot{x}^3o^3) - (ax^2 + 2a\dot{x}ox + a\dot{x}^2o^2)$$
$$+ (axy + a\dot{x}oy + a\dot{y}ox + a\dot{x}\dot{y}o^2) - (y^3 + 3\dot{y}oy^2 + 3\dot{y}^2o^2y + \dot{y}^3o^3) = 0.$$

Now by hypothesis $x^3 - ax^2 + axy - y^3 = 0$, and when these terms are erased and the rest divided by o there will remain

$$3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3o^2 - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3o^2 = 0.$$

But further, since o is supposed to be infinitely small so that it be able to express the moments of quantities, terms which have it as a factor will be equivalent to nothing in respect of the others. I therefore cast them out and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$.

It is accordingly to be observed that terms not multiplied by o will always vanish, as also those multiplied by o of more than one dimension; and that the remaining terms after division by o will always take on the form they should have according to the rule. This is what I wanted to show.

12.A6 Finding fluents from a fluxional relationship

Problem 1

When a fluent quantity is exhibited, the relationship of whose moments to those of some other fluent quantity is given, to find the relation of the quantities to one another.

Multiply the value of the ratio of the moments of the quantity sought to the moments of the exhibited quantity (so long as it is free from irrationals and not affected with some denominator of several terms) by the exhibited quantity, then divide each term individually by its own number of dimensions in this same quantity: what results will be the value of the quantity sought.