Some comments from me:

#1 This should be short and clean. There's nothing complicated here. The part about the numerator being constant is not required for the problem, as it is a statement not a directive, but it's nice to see it explained.

#2 I really think Bressoud must've intended this to be a different problem, but I honestly like this in the nasty way it is. It shows that mathematics complains when you go against physics (and constant angular velocity goes against physics). Do note that he explicitly writes equal area in equal time for #16-18, so I do not think that is an acceptable inference here without it being included. _If it were_, then the orbit would need to be circular, which simplifies everything drastically. Aside from this being ugly, there were two common mistakes, one easy to lose track of - do remember that, for example, when we differentiate r(2+cos ø) = 2 (oh, I type ø sometimes for theta just because I can on this computer easily, please don't be disturbed and please don't think I don't know the difference), both r and ø are functions of time. For many when taking the second derivative of r with respect to t, some of these extra derivatives were lost. Be careful with this. Don't forget how to take implicit derivatives (which is happening whenever you take any time derivative of anything other than a function of time). The other issue here is the technical one at the end. the ±√ is ugly and not a good solution. But without ± it is wrong, because the orbit surely travels to values where sin ø is positive and negative.

#3: I don't really have any comments. This should not have been difficult. It is good to realise what a "clean" simplified version would look like.

Some other comments to remember:

Your first Quizam is next Wednesday, 3 March. You will have 30 minutes to complete it. It will require familiarity with our techniques but not remembering equations. 30 minutes includes processing time - I strongly recommend stopping with 5 minutes to begin uploading. There is a practice quiz under "quizzes" in Canvas so you can be familiar with how Canvas will look. The quizam will be available from 12:01a on Wednesday 3 March to 11:59p on Thursday 4 March. You may _not_ access any materials while taking the quizam and you are prohibited from communicating with anyone about the quizam until you have completed it. Any voilation of these policies will result in failing the course and being reported for a violation of academic integrity.

Your second problem set is due Wednesday, 10 March. We will always have completed the material for the next PS before the prior one is due. Some have already completed it. You have no reason not to start.

We do not have a new lecture on Friday. Reading this and comparing your work to the exemplars is your course-task for Friday. On Monday I will review chapter 4 and at least begin to address any questions from chapters 5,6,7. Your chance to offer input for that is ending soon. I will record Monday's video very soon.

1.6: Exercise 4 Using the relationships of Equation (1.7), prove that

$$\frac{d\theta}{dt} = \frac{x(\frac{dy}{dt}) - y(\frac{dx}{dt})}{x^2 + y^2}$$

It follows that the acceleration is radial if and only if $x\frac{dy}{dt}$.

<u>**Proof:**</u> Using equations $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta$$

$$\implies \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta$$

$$\implies \theta = \arctan\left(\frac{y}{x}\right)$$

Since $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\frac{d\theta}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{y}{x}\right)$.

Let $\frac{y}{x} = u$. $\quad \frac{d}{dt}\tan^{-1}(u) = \frac{u'}{1+u^2}$ $\qquad u = \frac{y}{x} \quad \frac{du}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2+y^2} \checkmark$$

Since $\vec{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\vec{u}_\perp$,

$\vec{a}(t) = a\vec{u}_r$ iff $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\vec{u}_\perp = 0$

$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ when $r^2\frac{d\theta}{dt}$ is constant.

Since $r^2\frac{d\theta}{dt} = \frac{(x^2+y^2)\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)}{x^2+y^2} = x\frac{dy}{dt} - y\frac{dx}{dt}$,

the acceleration is exclusively radial

3.2

15. Consider a particle whose path is the ellipse

$$r(2+\cos\theta)=2$$

traversed in a counterclockwise direction about the origin, and that sweeps out one unit of area per unit time:

$$\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=1$$

Find the velocity and acceleration expressed in terms of the local coordinates $\vec{u}_r$ and $\vec{u}_\perp$.

By Equation (1.15),

$$\vec{V}=\frac{d\vec{r}}{dt}=\frac{dr}{dt}\vec{u}_r+r\frac{d\theta}{dt}\vec{u}_\perp$$

Find $\frac{d\theta}{dt}$

$$\frac{1}{2}r^2\frac{d\theta}{dt}=1$$

$$r^2\frac{d\theta}{dt}=2$$

$$\frac{d\theta}{dt}=\frac{2}{r^2}$$

Find $\frac{dr}{dt}$

$$r=\frac{2}{2+\cos\theta}=2(2+\cos\theta)^{-1}$$

$$\frac{dr}{dt}=-2(2+\cos\theta)^{-2}\cdot\left(-\sin\theta\cdot\frac{d\theta}{dt}\right)$$

CONT. →

$$\frac{dr}{dt} = \frac{-2\sin\theta}{(2+\cos\theta)^2} \cdot \frac{2}{r^2} = \frac{-1}{2+\cos\theta} \cdot \frac{2}{2+\cos\theta}(-\sin\theta) \cdot \frac{2}{r^2}$$

$$\frac{dr}{dt} = \frac{r}{-2} \cdot r \cdot (-\sin\theta) \cdot \frac{2}{r^2} = \frac{2r^2}{2r^2} \cdot \sin\theta = \sin\theta$$

$$\vec{V} = (\sin\theta)\vec{U}_r + r\left(\frac{2}{r^2}\right) U_\perp$$

$$\boxed{\vec{V} = (\sin\theta)\vec{U}_r + \left(\frac{2}{r}\right)\vec{U}_\perp}$$

Theorem 1.4 If a particle moves so that its acceleration is always radial and if the particle follows the curve $r(1+\varepsilon\cos\theta)=c$, then the acceleration is

$$\vec{a}(t) = \frac{-k^2}{cr^2}\vec{U}_r$$

where $k = 2(dA/dt)$ is a constant.

The particle does follow the curve with

$$\varepsilon = \frac{1}{2}, \quad c = 1$$

And acceleration is radial because $r^2 \frac{d\theta}{dt} = 2$ is a constant. Therefore,

$$\vec{a}(t) = \frac{-k^2}{cr^2}\vec{U}_r$$

Find k

$$k = 2\left(\frac{dA}{dt}\right) = 2(1) = 2$$

$$\vec{a}(t) = \frac{-(2)^2}{(1)r^2}\vec{U}_r = \frac{-4}{r^2}\vec{U}_r \qquad \boxed{\vec{a}(t) = \frac{-4}{r^2}\vec{U}_r}$$

Section 1.6 - Problem 19

Given: $\frac{d\theta}{dt} = k$ & $r(1+\varepsilon\cos\theta) = c$

Eqn(1.16) $\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta\right)$

Let: $\frac{d\theta}{dt} = \omega$ instead , $\frac{d^2\theta}{dt^2} = 0$ since ω is a constant.

$$r = \frac{c}{1+\varepsilon\cos\theta} \quad , \quad \frac{dr}{dt} = \frac{0 - \varepsilon(-\sin\theta)\frac{d\theta}{dt}\cdot c}{(1+\varepsilon\cos\theta)^2} = \frac{c\varepsilon\sin\theta\,\omega}{(1+\varepsilon\cos\theta)^2}$$

$$\frac{d^2r}{dt^2} = \frac{(c\varepsilon\omega^2\cos\theta)(1+\varepsilon\cos\theta)^{\cancel{2}} - (c\varepsilon\omega\sin\theta)2\cancel{(1+\varepsilon\cos\theta)}(-\varepsilon\omega\sin\theta)}{(1+\varepsilon\cos\theta)^{\cancel{4}3}}$$

$$= \frac{(1+\varepsilon\cos\theta)(c\varepsilon\omega^2\cos\theta) + 2(c\varepsilon\omega\sin\theta)^2}{(1+\varepsilon\cos\theta)^3} = \frac{\omega^2 c\varepsilon\cos\theta}{(1+\varepsilon\cos\theta)^2} + \frac{2(c\varepsilon\omega\sin\theta)^2}{(1+\varepsilon\cos\theta)^3}$$

$$\cos\theta = \frac{c}{\varepsilon r} - \frac{1}{\varepsilon} \qquad \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{(c-r)^2}{\varepsilon^2 r^2}$$

$$\Rightarrow \frac{d^2r}{dt^2} = \frac{\omega^2 c\varepsilon\left(\frac{c}{\varepsilon r} - \frac{1}{\varepsilon}\right)}{\left[1+\varepsilon\left(\frac{c}{\varepsilon r} - \frac{1}{\varepsilon}\right)\right]^2} + \frac{2c^2\varepsilon^2\omega^2\left[1 - \frac{(c-r)^2}{\varepsilon^2 r^2}\right]}{\left[1+\varepsilon\left(\frac{c}{\varepsilon r} - \frac{1}{\varepsilon}\right)\right]^3}$$

$$= \frac{\frac{\omega^2 c^2}{r} - \omega^2 c}{\left(\frac{c}{r}\right)^2} + \frac{2c^2\varepsilon^2\omega^2 - \frac{c^2\omega^2(c-r)^2}{r^2}}{\left(\frac{c}{r}\right)^3}$$

$$= \omega^2 r - \frac{\omega^2 r^2}{c} + \frac{2r^3\varepsilon^2\omega^2}{c} - \frac{\omega^2 r(c-r)^2}{c}$$

$$= \omega^2 r\left[1 - \frac{r}{c} + \frac{2(r\varepsilon)^2}{c} - \frac{(c-r)^2}{c}\right]$$

$$a_r: \quad \frac{d^2r}{dt^2} = \frac{\omega^2 r}{c}\left[2(r\varepsilon)^2 - (c-r)^2 - r + c\right]$$

$$2\frac{dr}{dt}\frac{d\theta}{dt} = 2\left(\frac{c\varepsilon\sin\theta\,\omega}{(1+\varepsilon\cos\theta)^2}\right)(\omega) = \frac{2\omega^2 c\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2} \qquad \sin\theta = \pm\sqrt{1 - \left(\frac{c-r}{\varepsilon r}\right)^2}$$

$$a_\theta: \quad = \pm\frac{2\omega^2 c\varepsilon\sqrt{1-\left(\frac{c-r}{\varepsilon r}\right)^2}}{\left(\frac{c}{r}\right)^2} = \pm\frac{2(\omega r)^2\varepsilon}{c}\sqrt{1-\left(\frac{c-r}{\varepsilon r}\right)^2}$$

Section 1.6 - Problem 19 (Continued)

$$\frac{d^2r}{dt^2} = \frac{\omega^2 r}{c}\left[2(r\varepsilon)^2 - (c-r)^2 - r + c\right]$$

$$2\frac{dr}{dt}\frac{d\theta}{dt} = \pm\frac{2(\omega r)^2\varepsilon}{c}\sqrt{1-\left(\frac{c-r}{\varepsilon r}\right)^2}$$

Then:

$$\boxed{\vec{a} = \frac{\omega^2 r}{c}\left[2(r\varepsilon)^2 - (c-r)^2 - r + c\right]\vec{u}_r \pm \left[\frac{2(\omega r)^2\varepsilon}{c}\sqrt{1-\left(\frac{c-r}{\varepsilon r}\right)^2}\right]\vec{u}_\theta}$$

Remark:

Notice the perpendicular component can either be positive or negative. This is telling us that the trajectory of an elliptical orbit follows radial conventions for when $\sin\theta$ is either positive or negative depending on the coordinate of the particle.

If this is not the case, the particle would be found to the upper orbit which breaks Celestial Mechanics.