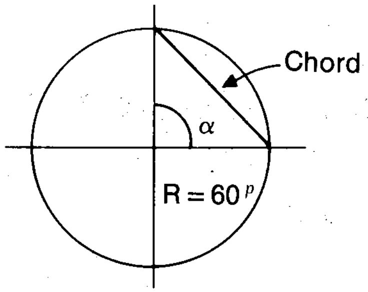
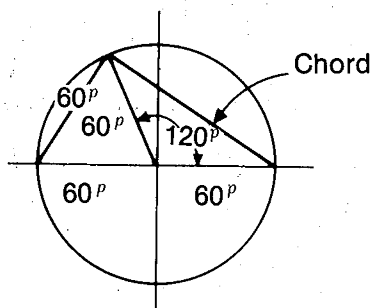


$$\text{Crd } 90^\circ = \sqrt{(60^p)^2 + (60^p)^2} = 60\sqrt{2}^{\,p} = 84^p\ 51'10''$$



$$\text{Crd } 120^\circ = \sqrt{(120^p)^2 - (60^p)^2} = 60\sqrt{3}^{\,p} = 103^p\ 55'23''$$

He also found Crd 36°, the side of a regular inscribed decagon, equals $30^p(\sqrt{5} - 1) = 37^p\ 4'\ 55''$. In Book IX of the *Almagest*, Ptolemy proved that for any arc S less than 180° $(\text{Crd } S)^2 + \text{Crd } (180 - S)^2 = (120^p)^2$. This is equivalent to the modern relationship in a unit circle $\sin^2\alpha + \cos^2\alpha = 1$, where α is any acute angle.

Ptolemy wrote widely on other sciences. In addition to his first work, *Almagest*, he authored in astronomy *Canobic Inscription* and *Handy Tables*, two systematic continuations of the *Almagest*, as well as *Tetrabiblos*, *Analemma*, and *Planetary Hypotheses*. In related fields he wrote *Optics*, which examines astronomical refraction, and *Harmonics*. Late in his career he wrote *Geography*, which places the prime meridian in the Canary Islands. In the 16th century, the map projections of the *Geography* stimulated the development of cartography.

44. From the *Syntaxis* or *Almagest* i*

(Trigonometry: Table of Sines)

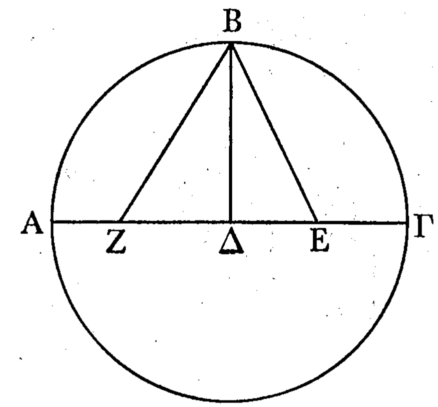
CLAUDIUS PTOLEMY

10. ON THE LENGTHS OF THE CHORDS IN A CIRCLE

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of these [chords in a circle], dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set out correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considerations.[1] In general we shall use the sexagesimal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

(ii.) *sin 18° and sin 36°*

32. First, let ABΓ be a semicircle on the diameter AΔΓ and with centre Δ, and from Δ let ΔB be drawn perpendicular to AΓ, and let ΔΓ be bisected at



E, and let EB be joined, and let EZ be placed equal to it, and let ZB be joined. I say that ZΔ is the side of a decagon, and BZ of a pentagon.[2]

For since the straight line ΔΓ is bisected at E, and the straight line ΔZ is added to it,

$$\Gamma Z \cdot Z\Delta + E\Delta^2 = EZ^2 \quad \text{[Eucl. ii. 6]}$$
$$= BE^2,$$

since EB = ZE.

But $E\Delta^2 + \Delta B^2 = EB^2$; [Eucl. i. 47]

therefore $\Gamma Z \cdot Z\Delta + E\Delta^2 = E\Delta^2 + \Delta B^2$.

When the common term $E\Delta^2$ is taken away,

the remainder $\Gamma Z \cdot Z\Delta = \Delta B^2$

i.e., $= \Delta\Gamma^2$;

therefore ZΓ is divided in extreme and mean ratio at Δ [Eucl. vi., Def. 3]. Therefore, since the side of the hexagon and the side of the decagon inscribed in the same circle when placed in one straight line are cut in extreme and

*Source: From *Selections Illustrating the History of Greek Mathematics*, trans. by Ivor Thomas (1941), vol. II, 413-425 with Greek originals deleted. Reprinted by permission of Harvard University Press.

mean ratio [Eucl. xiii. 9], and ΓΔ, being a radius, is equal to the side of the hexagon [Eucl. iv. 15, coroll.], therefore ΔZ is equal to the side of the decagon. Similarly, since the square on the side of the pentagon is equal to the rectangle contained by the side of the hexagon and the side of the decagon inscribed in the same circle [Eucl. xiii. 10], and in the right-angled triangle BΔZ the square on BZ is equal [Eucl. i. 47] to the sum of the squares on BΔ, which is a side of the hexagon, and ΔZ, which is a side of the decagon, therefore BZ is equal to the side of the pentagon.

Then since, as I said, we made the diameter[3] consist of 120^p, by what has been stated ΔE, being half of the radius, consists of 30^p and its square of 900^p, and BΔ, being the radius, consists of 60^p and its square of 3600^p, while EB^2, that is EZ^2, consists of 4500^p; therefore EZ is approximately $67^p\ 4'\ 55''$,[4] and the remainder ΔZ is $37^p\ 4'\ 55''$. Therefore the side of the decagon, subtending an arc of 36° (the whole circle consisting of 360°), is $37^p\ 4'\ 55''$ (the diameter being 120^p). Again, since ΔZ is $37^p\ 4'\ 55''$, its square is $1375^p\ 4'\ 15''$, and the square on ΔB is 3600^p, which added together make the square on BZ $4975^p\ 4'\ 15''$, so that BZ is approximately $70^p\ 32'\ 3''$. And therefore the side of the pentagon, subtending 72° (the circle consisting of 360°), is $70^p\ 32'\ 3''$ (the diameter being 120^p).

Hence it is clear that the side of the hexagon, subtending 60° and being equal to the radius, is 60^p. Similarly, since the square on the side of the square,[5] subtending 90°, is double of the square on the radius, and the square on the side of the triangle, subtending 120°, is three times the square on the radius, while the square on the radius is 3600^p, the square on the side of the square is 7200^p and the square on the side of the triangle is 10800^p. Therefore the chord subtending 90° is approximately $84^p\ 51'\ 10''$ (the diameter consisting of 120^p), and the chord subtending 120° is $103^p\ 55'\ 23''$.[6]

(iii.) *$\sin^2\theta + \cos^2\theta = 1$*

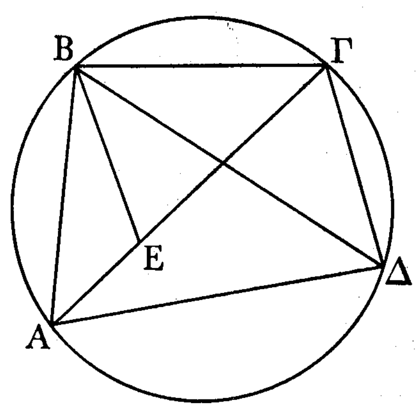
35. The lengths of these chords have thus been obtained immediately and by themselves,[7] and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending 36° was shown to be $37^p\ 4'\ 55''$ and its square $1375^p\ 4'\ 15''$, while the square on the diameter is 14400^p, therefore the square on the chord subtending the remaining 144° in the semicircle is $13024^p\ 55'\ 45''$ and the chord itself is approximately $114^p\ 7'\ 37''$, and similarly for the other chords.[8]

We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(iv.) *"Ptolemy's Theorem"*

36. Let ABΓΔ be any quadrilateral inscribed in a circle, and let AΓ and BΔ be joined. It is required to prove that the rectangle contained by AΓ and BΔ is equal to the sum of the rectangles contained by AB, ΔΓ and AΔ, BΓ.

For let the angle ABE be placed equal to the angle ΔBΓ. Then if we add the angle EBΔ to both, the angle ABΔ = the



angle EBΓ. But the angle BΔA = the angle BΓE [Eucl. iii. 21], for they subtend the same segment; therefore the triangle ABΔ is equiangular with the triangle BΓE.

∴ BΓ : ΓE = BΔ : ΔA; [Euclid. vi. 4]
∴ BΓ . AΔ = BΔ . ΓE. [Eucl. vi. 6]

Again, since the angle ABE is equal to the angle ΔBΓ, while the angle BAE is equal to the angle BΔΓ [Eucl. iii. 21], therefore the triangle ABE is equiangular with the triangle BΓΔ;

∴ BA : AE = BΔ : ΔΓ; [Eucl. vi. 4]
∴ BA . ΔΓ = BΔ . AE. [Eucl. vi. 6]

But it was shown that

BΓ . AΔ = BΔ . ΓE;

and ∴ AΓ . BΔ = AB . ΔΓ + AΔ . BΓ; [Eucl. ii. 1]

which was to be proved.

NOTES

1. . . . Ptolemy meant more than a graphical method; the phrase indicates a *rigorous proof* by means of geometrical considerations, as will be seen when the argument proceeds: It may be inferred, therefore, that when Hipparchus proved "by means of lines" (*On the Phaenomena of Eudoxus and Aratus*, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations; in other words, he was familiar with the main formulae of spherical trigonometry.

2. *i.e.*, ZΔ is equal to the side of a regular decagon, and BZ to the side of a regular pentagon, inscribed in the circle ABΓ.

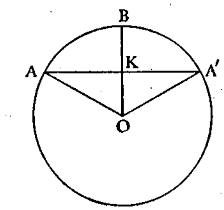
2. Following the usual practice, I shall denote *segments* of the diameter by ^{p}, sixtieth parts of a τμῆμα by the numeral with a single accent, and second-sixtieths by the numeral with two accents. As the circular associations of the system tend to be forgotten, and it is used as a general system of enumeration, the same notation will be used for the *squares* of parts.

4. Theon's proof that $\sqrt{4500}$ is approximately $67^p\ 4'\ 55''$ has already been given (vol. i. pp. 56-61).

5. This is, of course, the square itself; the Greek phrase is not so difficult. We could translate, "the second power of the side of the square," but the notion of powers was outside the ken of the Greek mathematician.

6. Let AB be a chord of a circle subtending an angle α at the centre O, and let AKA′ be drawn perpendicular to OB so as to meet OB in K and the circle again in A′. Then

$$\sin\alpha\,(=\sin AB) = \frac{AK}{AO} = \frac{\tfrac{1}{2}AA'}{AO}.$$



And AA′ is the chord subtended by double of the arc AB, while Ptolemy expresses the lengths of chords as so many 120th parts of the diameter; therefore sin α is half the chord subtended by an angle 2α at the centre, which is conveniently abbreviated by Heath to ½(crd. 2α), or, as we may alternatively express the relationship, sin AB is "half the chord subtended by double of the arc AB," which is the Ptolemaic form; as Ptolemy means by this expression precisely what we mean by sin AB, I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that cos α [$=\sin(90-\alpha)$] = ½ crd. $(180° - 2\alpha)$, or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." Tan α and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended Ptolemy proves that

side of decagon (=crd. 36°=2 sin 18°)=$37^p\ 4'\ 55''$,
side of pentagon (=crd. 72°=2 sin 36°)=$70^p\ 32'\ 3''$,
side of hexagon (=crd. 60°=2 sin 30°)=60^p,
side of square (=crd. 90°=2 sin 45°)=$84^p\ 51'\ 10''$,
side of equilateral triangle (=crd. 120°=2 sin 60°)=$103^p\ 55'\ 23''$.

7. *i.e.*, not deduced from other known chords.

8. *i.e.*, crd. 144°(=2 sin 72°)=$114^p\ 7'\ 37''$. If the given chord subtends an angle 2θ at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle $(180 - 2\theta)$, and the theorem asserts that

$$(\text{crd. } 2\theta)^2 + (\text{crd. } \overline{180-2\theta})^2 = (\text{diameter})^2,$$

or $\sin^2\theta + \cos^2\theta = 1$.