44. From the Syntaxis or Almagest

(Trigonometry: Table of Sines)

CLAUDIUS PTOLEMY

10. ON THE LENGTHS OF THE CHORDS IN A CIRCLE

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of those chords in a circle, dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considerations. In general we shall use the hexagonal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

In trigonometry, the sine of an angle is equal to the ratio of the opposite side to the hypotenuse in a right-angled triangle.

\[ \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \]

32. First, let \( \Delta \) be a semicircle on the diameter \( \Delta \Delta \) and with centre \( \Delta \), and from \( \Delta \) let \( \Delta \beta \) be drawn perpendicular to \( \Delta \Delta \), and let \( \Delta \Delta \) be bisected at \( \varepsilon \), and let \( \varepsilon \varepsilon \) be joined, and let \( \varepsilon \varepsilon \) be placed equal to it, and let \( \varepsilon \beta \) be joined.

I say that \( \varepsilon \varepsilon \) is the side of a decagon, and \( \varepsilon \beta \) of a pentagon.

For since the straight line \( \Delta \Delta \) is bisected at \( \varepsilon \), and the straight line \( \Delta \varepsilon \) is added to it,

\[ \varepsilon \varepsilon + \varepsilon \beta = \varepsilon \beta \]

But since \( \varepsilon \varepsilon \beta \Delta \) is a right-angled triangle (Eucl. ii. 11),

\[ \varepsilon \varepsilon = \varepsilon \beta + \Delta \beta \]  

Therefore \( \varepsilon \varepsilon = \varepsilon \beta = \varepsilon \beta + \Delta \beta \).

When the common term \( \varepsilon \beta \) is taken away, the remainder \( \varepsilon \varepsilon = \Delta \beta \) (i.e., \( \Delta \beta = \varepsilon \varepsilon \)).

Therefore \( \Delta \varepsilon \) is divided in extreme and mean ratio at \( \Delta \) (Eucl. vi., Def. 3). Therefore, since the side of the hexagon and the side of the decagon inscribed in the same circle when placed in one straight line are cut in extreme and
mean ratio (Eucl. xiii. 9), and if \( \Gamma A \), being a radius, is equal to the side of the hexagon (Eucl. iv. 15, coroll.), therefore \( \Delta Z \) is equal to the side of the decagon. Similarly, since the square on the side of the pentagon is equal to the rectangle contained by the side of the hexagon and the side of the decagon inscribed in the same circle (Eucl. xiii. 10), and in the right-angled triangle \( \triangle B Z \), the square on \( B Z \) is equal (Eucl. i. 47) to the sum of the squares on \( B A \) which is a side of the hexagon, and \( \Delta E \), which is a side of the decagon, therefore \( B Z \) is equal to the side of the pentagon. Then since, as I said, we made the diameter of a circle consist of 120°, by what has been stated above, being half of the radius, consists of 30° and its square of 900°, and \( B A \), being the radius, consists of 60° and its square of 3600°, while the square on \( B Z \), that is \( \Delta E \), consists of 4500°; therefore \( B Z \) is approximately \( 60^\circ 4' 55'' \) and the remainder \( \Delta Z \) is \( 37^\circ 4' 55'' \). Therefore the side of the decagon, subtending an arc of 36° the whole circle consisting of 360°, is \( 37^\circ 4' 55'' \) the diameter being 120°. Again, since \( \Delta A \) is \( 37^\circ 4' 55'' \), its square is \( 13755^\circ 4' 15'' \), then the square on \( AB \) is \( 3600\times \frac{13755}{120^\circ} = 13755^\circ 4' 15'' \), so that \( B Z \) is approximately \( 37^\circ 32' 32'' \). And therefore the side of the pentagon, subtending \( 32^\circ \) the circle consisting of 360°, is \( 37^\circ 32' 32'' \) the diameter being 120°.

Hence it is clear that the side of the hexagon, subtending 60° and being equal to the radius, is \( 60^\circ \). Similarly, since the square on the side of the square, 90°, subtending 90°, is double of the square on the radius, and the square on the side of the triangle, subtending 120°, is three times the square on the radius, while the square on the radius is 3600°, the square on the side of the square is 7200° and the square on the side of the triangle is 10800°. Therefore the chord subtending 90° is approximately \( 4^\circ 51' 10'' \) the diameter consisting of 120°, and the chord subtending 120° is \( 103^\circ 55' 23'' \).

(3.3) sin \( \theta \) + cos \( \theta \) - 1 = 1

35. The lengths of these chords have thus been obtained immediately and by themselves; and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending 36° was shown to be \( 37^\circ 4' 55'' \) and its square is \( 13755^\circ 4' 15'' \), while the square on the diameter is \( 14400^\circ \), therefore the square on the chord subtending the remaining 144° in the semicircle is \( 13024^\circ 55' 45'' \) and the chord itself is approximately \( 114^\circ 7' 37'' \), and similarly for the other chords.

We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(3.4) "Ptolemy's Theorem."

36. Let \( A\Gamma B\) be any quadrilateral inscribed in a circle, and let \( \Delta A \) and \( B\Gamma \) be joined. It is required to prove that the rectangle contained by \( \Delta A \) and \( B\Gamma \) is equal to the sum of the rectangles contained by \( A\Gamma \), \( \Delta A \), \( B\Gamma \), and \( \Delta B \).

For let the angle \( A\Gamma B \) be placed equal to the angle \( A\Gamma B \). Then if we add the angle \( E\Gamma B \) to both, the angle \( A\Gamma B \) = the angle \( E\Gamma B \). But the angle \( B\Gamma A \) = the angle \( B\Gamma E \) (Eucl. iii. 21), for they subtend the same segments; therefore the triangle \( A\Gamma B \) is equiangular with the triangle \( B\Gamma E \).

\[
\begin{align*}
&\text{If } \angle B = \angle B', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle A = \angle A', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle C = \angle C', \text{ then } \triangle ABC \cong \triangle A'B'C'.
\end{align*}
\]

Again, since the angle \( \Delta B \) is equal to the angle \( \Delta B' \), while the angle \( \Delta B' \) is equal to the angle \( \Delta B \) (Eucl. iii. 21), therefore the triangle \( \Delta B \) is equiangular with the triangle \( \Delta B' \).

\[
\begin{align*}
&\text{If } \angle B = \angle B', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle A = \angle A', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle C = \angle C', \text{ then } \triangle ABC \cong \triangle A'B'C'.
\end{align*}
\]

But it was shown that \( \text{If } \angle B = \angle B', \text{ then } \angle \Gamma A \) = \( \angle \Gamma A' \).

\[
\begin{align*}
&\text{If } \angle B = \angle B', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle A = \angle A', \text{ then } \triangle ABC \cong \triangle A'B'C'. \\
&\text{If } \angle C = \angle C', \text{ then } \triangle ABC \cong \triangle A'B'C'.
\end{align*}
\]

And \( \Delta A' \) is the chord subtended by double of the arc \( \Delta A \), while Ptolemy expresses the lengths of chords as so many 120° parts of the diameter; therefore since \( \sin \alpha \) is half the chord subtended by an angle 2\( \alpha \) in the circle, it is conveniently abbreviated by \( \text{Helioc.} \) 2\( \alpha \), or, as we may alternatively express the relationship, \( \sin \alpha \) is "half the chord subtended by double of the arc \( \alpha \)" which is the Ptolemaic form, as Ptolemy means by this expression precisely what we mean by \( \sin \alpha \), I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that \( \cos \alpha = \sqrt{1-\sin^2 \alpha} = \frac{\text{Helioc.} \sqrt{1-\sin^2 \alpha}}{2} \), or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." Tan \( \alpha \) and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended, Ptolemy proves that

\[
\begin{align*}
&\text{side of decagon } \equiv \text{side of pentagon } \equiv \text{side of } 120^\circ \equiv \text{side of } 60^\circ \equiv 103^\circ 55' 23''.
\end{align*}
\]

7. I.e., not deduced from other known chords.

8. I.e., \( 144^\circ = \text{side of pentagon } \equiv \text{side of } 120^\circ \equiv \text{side of } 60^\circ \equiv 103^\circ 55' 23'' \).

If the given chord subtends an angle \( 2\alpha \) at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle \( 180^\circ - 2\alpha \), and the theorem asserts that

\[
\begin{align*}
&\text{side of } 180^\circ - 2\alpha = \text{side of pentagon } \equiv \text{side of } 120^\circ \equiv \text{side of } 60^\circ \equiv 103^\circ 55' 23''.
\end{align*}
\]

\[
\begin{align*}
&\text{side of } 180^\circ - 2\alpha = \text{side of pentagon } \equiv \text{side of } 120^\circ \equiv \text{side of } 60^\circ \equiv 103^\circ 55' 23''.
\end{align*}
\]