# Object Integral: Charge

Consider an object with charge **non-uniformly** spread over it. How might you find the total charge on it?

Imagine plastic rod with charge distributed, and unevenly at that. Suppose that you have a 20cm long rod, with a charge density of $\lambda_1 = 2\,\mu\text{C/m}$ and one end, $\lambda_2 = 5\,\mu\text{C/m}$ at the other end, and smoothly varying in between.

### Pieces

With line-like objects, there is only one sensible way to do this. Imagine all the pieces to be the same size. We'll imagine the pieces to be so short, that the charge density is constant on each one.

### Axes

To describe this in algebraic form, we need a coordinate system that we can use to refer to different parts of the object. I choose the origin to be at the less charged end, and the x-axis to lie along the rod. Each bit is *dx* long.

### Contribution

Each little bit holds a charge $dq = \lambda\,dx = \lambda(x)\,dx$. Notice one differential per side.

### Express

Need to express λ (which is not a constant) in terms of the coordinates.
$\lambda = \left(2\,\mu\text{C/m}\right) + \left(15\,\mu\text{C/m}^2\right)x$ Test this by plugging in limiting values for *x*. Note how the units work. So now $dq = \left[\left(2\,\mu\text{C/m}\right) + \left(15\,\mu\text{C/m}^2\right)x\right]dx$

### Integrate

$$\int_{\text{rod}} dq = \int_{0\text{m}}^{0.2\text{m}} \left(2\,\mu\text{C/m}\right) + \left(15\,\mu\text{C/m}^2\right)x\;dx$$

$$q = \left(\left(2\,\mu\text{C/m}\right)x + \tfrac{1}{2}\left(15\,\mu\text{C/m}^2\right)x^2\right]_{0\text{m}}^{0.2\text{m}}$$

$$q = 0.7\,\mu\text{C}$$

Note that the integration step adds the integral sign, but does NOT involve adding the *dx*. It should already be there from a previous step.

Another Note: the integral limits **must** be increasing. The *dx* implies that direction.

# Object Integral: E Field Off End of Rod

Take a rod with uniform charge density λ and length $L$, and suppose we want to know the electric field a distance $a$ from the left end.

Follow the same procedure. Chop into **pieces**. **Axes** hint: choose to make one axis pass through the point P. Most often, this is a good plan.

**Contribution** for one piece: $dE = k\frac{dq}{x^2}$. Notice that $x$ is not $L$ or $a$. $x$ is variable, range $a$ to $a+L$. Notice also that we are working with the field magnitude: the component is of course in the negative direction.

Even though the charge is evenly spread, so λ is the same everywhere, still the different $dq$ will contribute differently to $E$ because they are different distances away. So that's why you need to find the contribution of each piece.

Even though Electric field is a vector, in this case all the vectors point in the same direction. So the equation above is already for the horizontal **component** of the field. With the standard equation $dq = \lambda\, dx$, it is also already **expressed** in terms of constants and coordinates

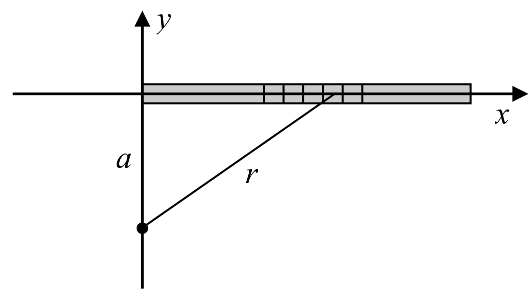
Then you integrate: $E = \int_a^{a+L} k\frac{\lambda\, dx}{x^2} = k\lambda\left(-\frac{1}{x}\right]_a^{a+L} = k\lambda\left(\frac{1}{a} - \frac{1}{a+L}\right) = \frac{k\lambda L}{a(a+L)}$

# Object Integral: E Field at Perpendicular off a Rod End

Set up with rod length $L$ and point P at distance $a$. Suppose that the charge density $\lambda$ is negative.

**Overall direction:**

Thinking of lots of little vectors, we can tell that the total field will be up and to the right. We handle this by treating each component ***separately***. Let us tackle the component parallel to the rod.




**Pieces**

Same as for all rod problems.

**Axes**

Definitely want an axis along the object. Small part is $dx$. Easiest if other axis is through point.

**Contribution**

A small ***generic*** piece still has charge $dq = \lambda\, dx$, and is small enough to act like a point charge.

- 1st, get the (small) field magnitude: $|d\mathbf{E}| = k\dfrac{|dq|}{r^2}$

**Components**

- Then take the component: $d\mathbf{E}_x = k\dfrac{|dq|}{r^2}\dfrac{x}{r}$

**Express**

First, get the differential in terms of a coordinate: $d\mathbf{E}_x = k\dfrac{|\lambda|\, dx}{r^3}x$ What else changes as you vary $x$ (i.e., choose different pieces)? $r = \sqrt{x^2 + a^2}$ So $d\mathbf{E}_x = k|\lambda|\dfrac{x\, dx}{\left(a^2 + x^2\right)^{3/2}}$.

**Integrate**

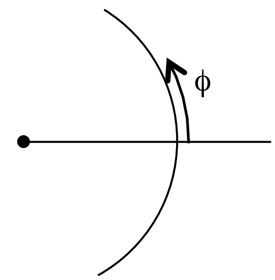
Recall: Only add integral sign, $dx$ already there. Ew! Nasty integral. Rely on Appendix or something.

$$\mathbf{E}_x = \int_0^L k|\lambda|\frac{x\, dx}{\left(a^2 + x^2\right)^{3/2}} = k|\lambda|\left(-\frac{1}{\left(a^2 + x^2\right)^{1/2}}\right]_0^L$$

$$= k|\lambda|\left(\frac{-1}{\sqrt{a^2 + L^2}} - \frac{-1}{\sqrt{a^2}}\right) = k|\lambda|\left(\frac{1}{a} - \frac{1}{\sqrt{a^2 + L^2}}\right)$$

# Object Integrals: E Field at Center of Arc

Consider the E-field at the center of curvature of a 120° arc, radius $R$ and with total charge $Q$. HRW show how to get the field at the "center" of the arc.




**Pieces:**

Still cut up object along its length. Still equal sized pieces.

**Axes**

In order to lay a coordinate axis along to the object, we have to introduce **circular coordinates** to this problem.

Call the length of the small pieces **arc length** $ds$. We can still use $dq = \lambda\, ds$. But then we'll convert using $\Delta s = R\,\Delta\phi \rightarrow ds = R\,d\phi$ (where the angle MUST be in radians). So we'll ultimately be chopping the ring into $d\phi$ pieces.

It is also important to choose where $\phi = 0$. The $\phi = 0$ point gets involved in two ways: how you take vector components, and limits of the integral. These two MUST agree!

**Contribution:**

$dE = k\dfrac{dq}{r^2}$ … and that's pretty much it!

**Components:**

Symmetry: Field points towards or away from arc middle, so vertical component is zero.

$$dE_x = -k\frac{dq}{r^2}\cos\phi$$

**Express:**

The only thing to fix is $dq$, but that has $\lambda$ instead of $Q$. Need $\lambda = \dfrac{Q}{L} = \dfrac{Q}{2\pi R/3}$.

$$dE_x = -k\frac{1}{r^2}\cos\phi\;\lambda r\,d\phi = -k\frac{1}{R^2}\cos\phi\;\frac{3Q}{2\pi R}R\,d\phi$$

**Integrate:**

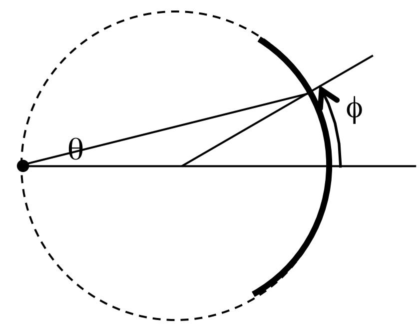
Notice how we have already made a choice about the coordinates, specifically where $\phi = 0$ lies.

$$E_x = -k\frac{1}{R^2}\frac{3Q}{2\pi}\int_{-\pi/3}^{\pi/3}\cos\phi\,d\phi$$

$$E_x = -k\frac{1}{R^2}\frac{3Q}{2\pi}\Big(\sin\phi\Big]_{-\pi/3}^{\pi/3} = -k\frac{1}{R^2}\frac{3Q}{2\pi}\left(\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= -k\frac{1}{R^2}\frac{3\sqrt{3}Q}{2\pi}$$

# Object Integral: E Field at Opposite Side of Arc




It turns out that you can also find the field a distance 2R from the arc center (on the other side of the completing circle).

**Pieces & Axes:**

Choose circular, origin still at the arc center. NEVER put the origin anywhere else! $dq = \lambda\, ds = \lambda R\, d\phi$

Choose $\phi = 0$ in the middle again. Symmetry, you know.

**Contribution & Component:**

Symmetry: Field points towards or away from arc center, so vertical component is zero.

$$dE = k\frac{dq}{r^2} \qquad dE_x = -k\frac{dq}{r^2}\cos\theta$$

**Express:**

Our standard $dq$ has the issue that λ is not given. $\lambda = \frac{Q}{L} = \frac{Q}{2\pi R/3}$, so $dq = \lambda R\, d\phi = \frac{3Q}{2\pi}d\phi$

There is a funky circle rule that says that $\theta = \frac{1}{2}\phi$.

There is a more funky circle rule that says $r$ and $2R$ are sides of a right triangle, so $\cos\theta = \frac{r}{2R}$.

Put it all together: $dE_x = -k\frac{1}{(2R\cos\theta)^2}\cos\theta\left(\frac{3Q}{2\pi}d\phi\right) = -k\frac{3Q}{8\pi R^2\cos(\phi/2)}d\phi$

**Integrate:**

Notice how we have already made a choice about the coordinates, specifically where $\phi = 0$ lies. We **must** abide by that now, in choosing integral limits.

$$E_x = -k\frac{3Q}{8\pi R^2}\int_{-\pi/3}^{\pi/3}\frac{1}{\cos(\phi/2)}\,d\phi$$

Barring a fancy calculator, the best way to evaluate this integral is with a change of variables.

$$E_x = -k\frac{3Q}{8\pi R^2}\int_{-\pi/6}^{\pi/6}\frac{1}{\cos\theta}\,2\,d\theta = -k\frac{3Q}{8\pi R^2}\Big(2\ln(\sec\theta+\tan\theta)\Big]_{-\pi/6}^{\pi/6}$$

$$E_x = -k\frac{3Q}{8\pi R^2}\left(2\ln 3\right)$$

# Object Integral: E Field on Ring Axis

Ring of radius $R$ and total charge $Q$, we want to know E field on the axis of the ring, and distance $z$ above its plane.

**Pieces & Axes:**

Still cut up object along its length. Still equal sized pieces. Choose circular coords, origin at the arc center, $dq = \lambda\, ds = \lambda R\, d\phi$

**Contribution & Components:**

Symmetry tells us that only the z-component is non-zero. Use symmetry to identify cancelation whenever possible.

One little piece causes a little field at our point: $dE = k\frac{dq}{r^2}$, $dE_z = k\frac{dq}{r^2}\frac{z}{r}$. NOTE: $r$ here is the distance from $dq$ to the point of interest; $r$ is ***NOT*** a radial distance from the center of the ring.

**Express:**

In terms of the coordinates and given constants. In this case, $r$ is actually constant, so we can just leave it. We do need to express λ in terms of the given $Q$.

$$dE_z = kz\frac{dq}{r^3} = kz\frac{\lambda R}{r^3}d\phi \qquad Q = \lambda\, 2\pi R \qquad dE_z = kz\frac{\frac{Q}{2\pi R}R}{r^3}d\phi = \frac{kzQ}{2\pi r^3}d\phi$$

**Integrate:**

Hey! Nothing in there varies! So we get an easy integral <<review getting limits>>

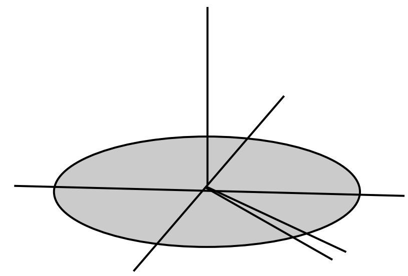
$$E_z = \frac{kQz}{r^3}$$

You might need to calculate $r$ from $R$ and $z$ if those were given in the problem statement. However, as far as the integration is concerned, that doesn't matter.

# Object Integral: E Field on Axis of a Disk of Charge

## *Wedge Method (HRW show Ring Method)*

If charge is spread over an *area*, then we of course describe it with a σ. In this class, we will *always* approach **surface** charge by chopping it into **linear** charged pieces for which we already have an answer.



Charged disk of radius $R$, surface charge density σ.

**Pieces & Axes:**

To do this differently from HRW, slice up into very thin wedges. Equal sized pieces as always. Cylindrical coords are still appropriate. However, we need a coordinate to measure radially from the $z$ axis; traditionally, that would be $r$ or ρ, but we want to reserve $r$ for Coulomb's law, and ρ for volume charge density. I will call the radial coordinate $r_x$, so that our coordinate system is $z$, $r_x$, $\phi$.

**Contribution & Components:**

The contribution is now the vertical field from a wedge. NOTE that this is NOT a point charge formula!! For doing a surface charge (with the method in this class), the $dQ$ will always be a long thin object, and you'll need a subsidiary integration problem. Sometimes it will already have been done, but not this time. (NOTE: I'm using $dQ$ for the charge on the thin wedge, and $dq$ for the charge on a small piece along a wedge.)

## *E Field Perpendicular off Tip of Thin Wedge*

The wedge is like a rod, but with more charge per length as you go away from the center: $\lambda = \lambda_{\max}\frac{r_x}{R}$. So do that problem: Very similar to the rod problem from before. We need a coordinate to measure

$$dE = k\frac{dq}{\left(\sqrt{r_x^2+z^2}\right)^2},\ dE_z = dE\frac{z}{\sqrt{r_x^2+z^2}} = k\frac{z\,dq}{\left(r_x^2+z^2\right)^{3/2}} = k\frac{z}{\left(r_x^2+z^2\right)^{3/2}}\frac{\lambda_{\max}r_x}{R}dr_x$$

$$E = \frac{k\lambda_{\max}z}{R}\int_0^R \frac{r_x}{\left(r_x^2+z^2\right)^{3/2}}dr_x = \frac{k\lambda_{\max}z}{R}\left(\frac{-1}{\sqrt{r_x^2+z^2}}\right]_0^R = \frac{k\lambda_{\max}z}{R}\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)$$

Back to the original problem. We have a wedge that is super thin, only $d\phi$ wide. Relate that to the very small $dQ$? Take a small patch of the disk near the edge, only $dr_x$ by $ds = R\,d\phi$ in size. It will have charge equal to $\lambda_{\max}dr_x = dq = \sigma\,dr_x\,Rd\phi$, so $\lambda_{\max} = \sigma R\,d\phi$. Thus, the small electric field produced by the very thin wedge is (based on the wedge formula)

$dE = \frac{kz\sigma R\,d\phi}{R}\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right) = kz\sigma\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)d\phi$ . No components necessary.

**Express:** No re-expressing necessary, it is already all given constants and coordinates:

**Integrate:**

Easier than it looks: mostly a constant: $E = kz\sigma\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)\int_0^{2\pi} d\phi = 2\pi kz\sigma\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)$