# Springs/Oscillations/Fluids

# Springs

From Hooke's law, it can be seen that springs are written in the form of ordinary differential equations who's solutions are in forms:

$$x(t) = A\cos(\omega t + \varphi) \qquad \text{or} \qquad x(t) = Ae^{i\omega t}$$

Where $A = Amplitude\ of\ Oscillations$ and $\varphi = Phase\ Angle$

A and ϕ are the two constants required to satisfy the constants of integration

The Potential Energy $U = \frac{1}{2}kx^2$ can also be incredibly useful for avoiding fully solving the differential equations

GRE book recommends approaching spring problems as follows:

- Limiting cases, dimensional arguments, and symmetry
- Conservation of Energy
- Writing and Solving the differential equation

# Dampened Oscillators

We can add some damping force to the system, generally air resistance and drag appear proportional to velocity $F = -kx$, where the EOM for the oscillator becomes

$$m\ddot{x} + b\dot{x} + kx = 0$$

There are three types of solutions for these systems: Underdamped, Overdamped, and Critically damped, generally represented as $\beta^2 < \omega_o^2$, $\beta^2 > \omega_o^2$, and $\beta^2 = \omega_o^2$, respectively

Underdamped solution: $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

This represents oscillations with exponential decay.

In overdamped situations, no actual oscillations occur, the oscillator exponentially returns to equilibrium

In critically damped situations, the system quickly returns to equilibrium

# Driven Oscillator and Circuit Analog

We can also add a driving force to the system:

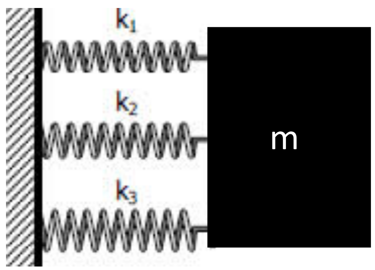
$$m\ddot{x} + b\dot{x} + kx = A\cos(\omega t)$$

The amplitude of oscillation is maximized when driving frequency equals resonant frequency $\omega_R^2 = \omega_o^2 - 2\beta^2$

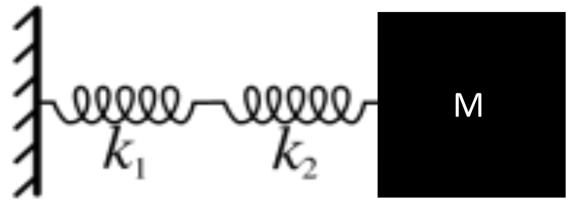
A circuit system is comparable to a mechanical system

| Mechanical System | | Circuit System | |
|---|---|---|---|
| x | displacement | q | charge |
| $\dot{x}$ | velocity | I | current |
| m | mass | L | inductance |
| b | damping resistance | R | resistance |
| 1/k | spring stiffness | C | capacitance |
| F | driving force amplitude | V | driving voltage amplitude |

# Problem




$$k_{eq} = k + k + k = 3k$$




$$k_{eq} = \left(\frac{1}{k} + \frac{1}{k}\right)^{-1} = \frac{k}{2}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{\sqrt{k/2M}}{\sqrt{3k/m}} = \sqrt{\frac{m}{6M}}$$

# Fluids

Bernoulli's principle:

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = Constant$$

So for pipes of different diameters

$$v_1 a^2 = v_2 b^2$$

$$\frac{v_1^2}{2} + gz_1 + \frac{p_1}{\rho} = \frac{v_2^2}{2} + gz_2 + \frac{p_2}{\rho}$$

$$p_2 = \frac{\rho v_1^2}{2}\left(1 - \frac{a^4}{b^4}\right) + p_1$$

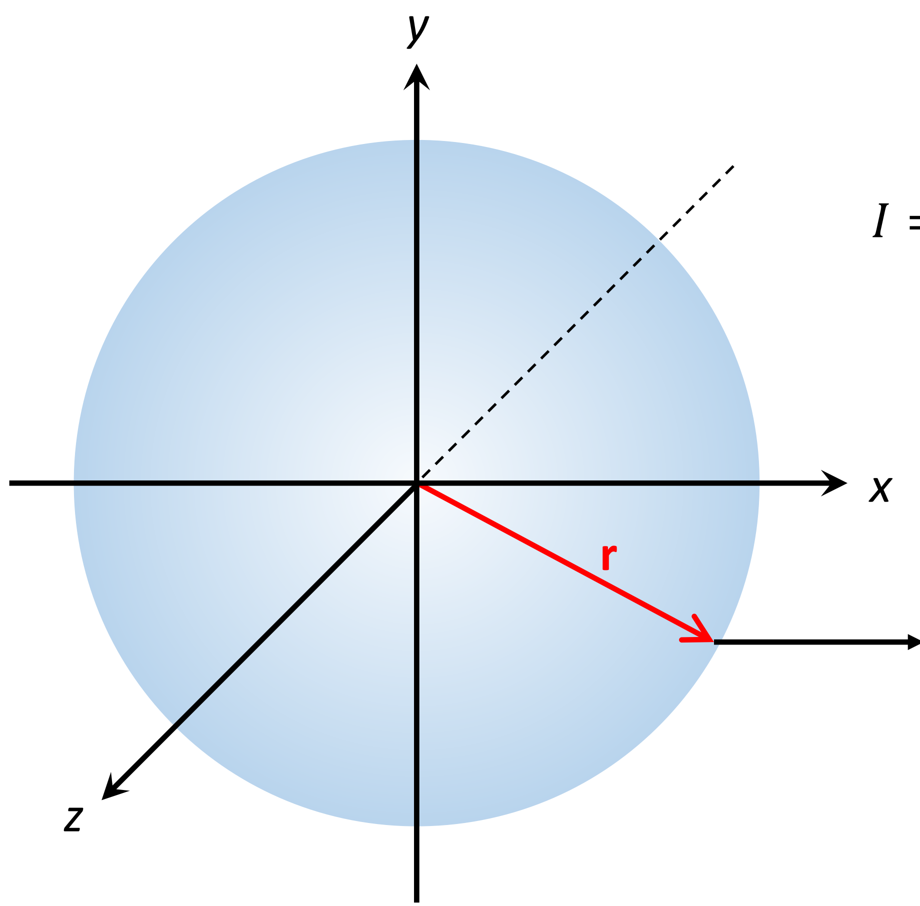
# Buoyant Force

$$F = \rho V g$$

# Moment of Inertia

First simple example:

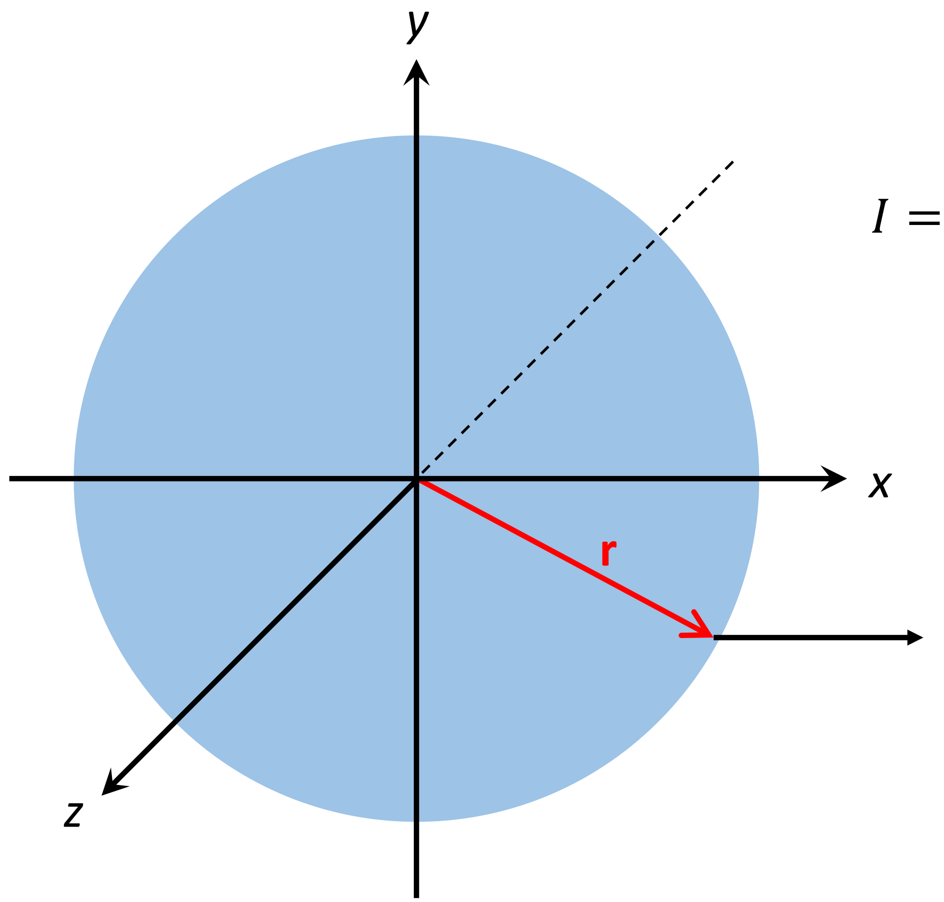
Rotation of a thin spherical shell, with radius *r*, about an axis through the center




$$I = \frac{2}{3}mr^2$$

First simple example:

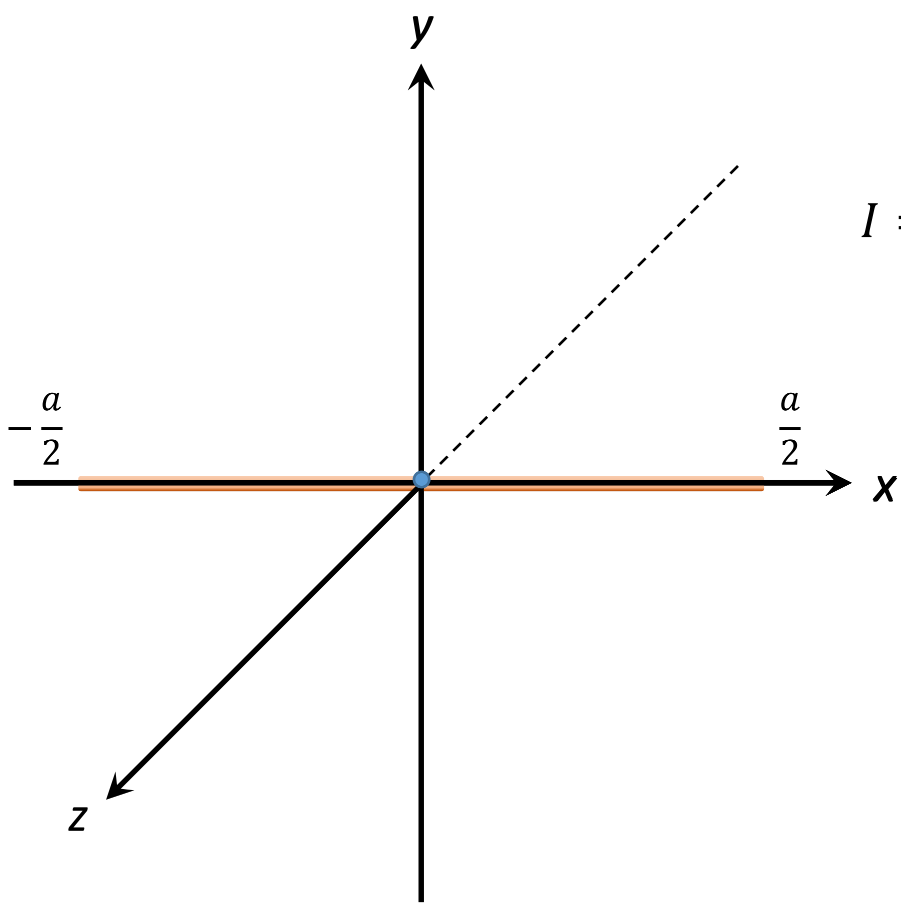
Rotation of a sphere, with radius *r*, about an axis through the center




$$I = \frac{2}{5} m r^2$$

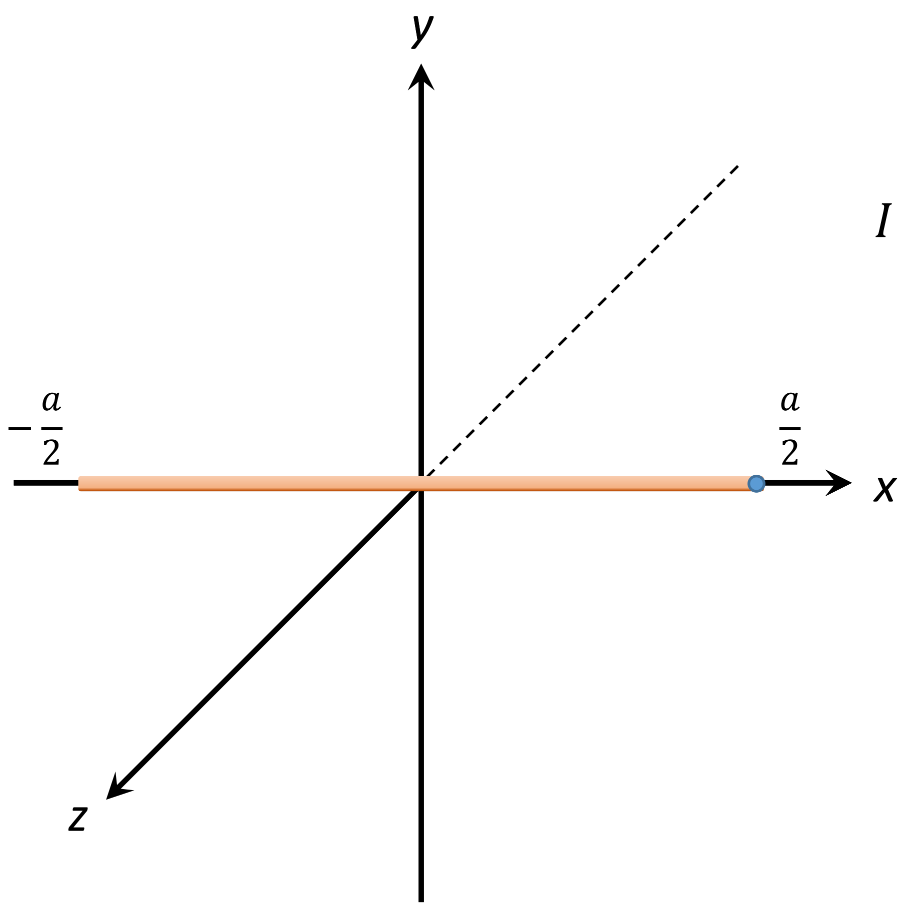
Second Example:

A uniform rod of length *a* and negligible thickness rotating about its center

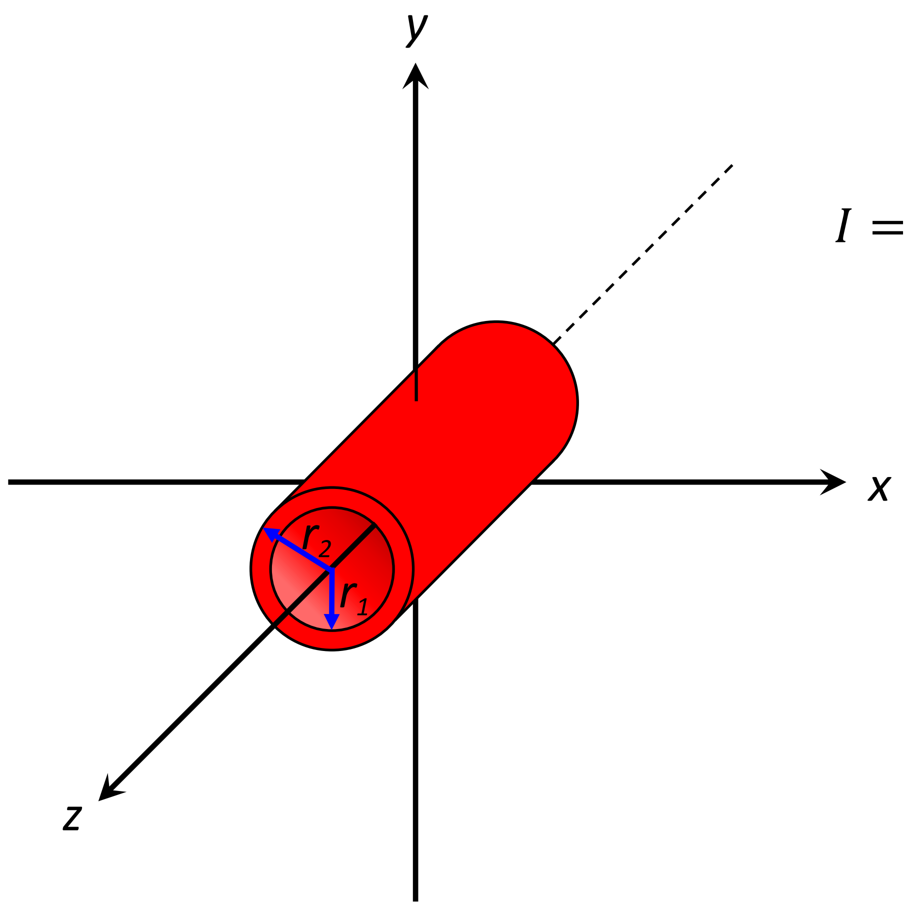



$$I = \frac{1}{12}ma^2$$

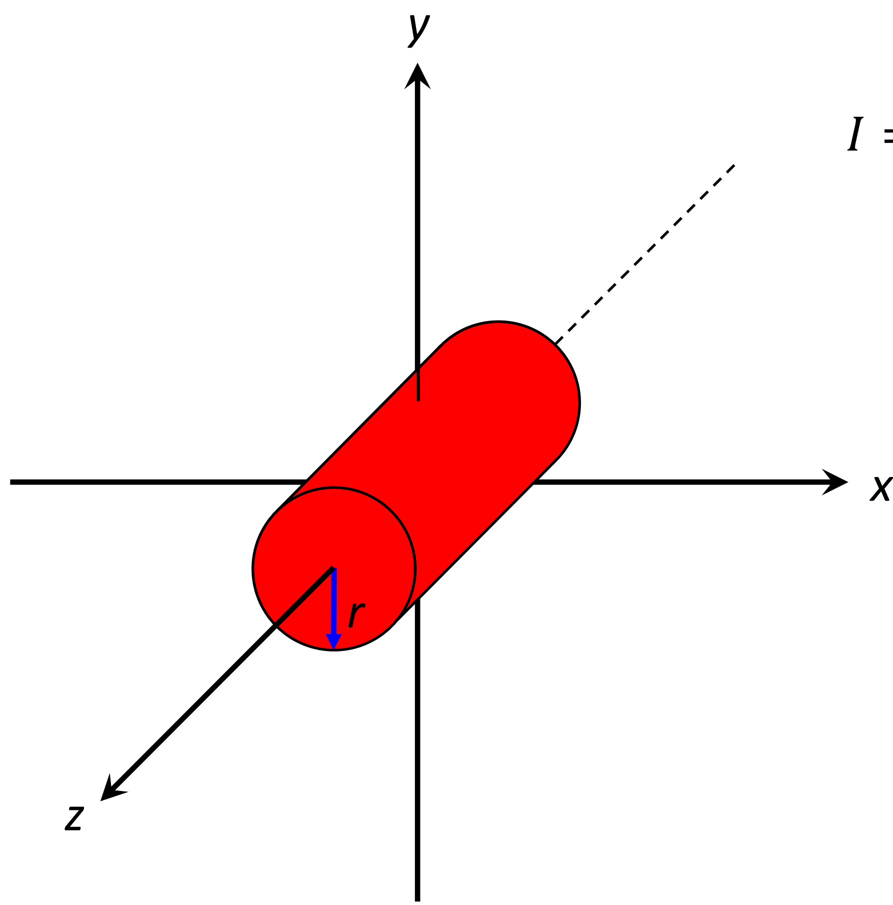
Third simple example:
Rotation of a rod with length *a* about its endpoint




$$I = \frac{1}{3}ma^2$$

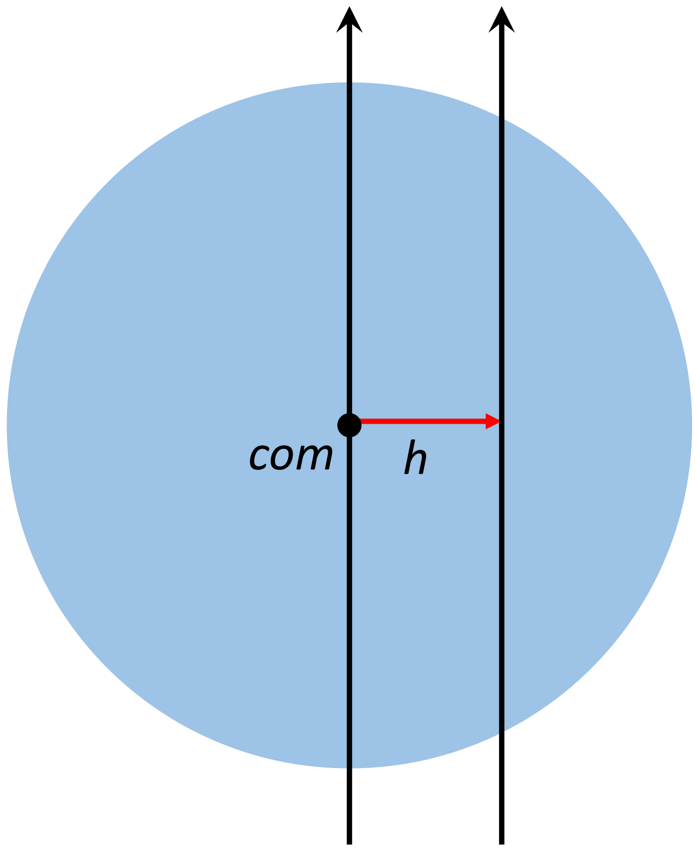



$$I = \frac{1}{12}m[r_1{}^2 + r_2{}^2]$$




$$I = \frac{1}{2} m r^2$$

Parallel Axis Theorem




$$I_{new} = I_{com} + mh^2$$

# Orbital Motion/Centripetal Motion/Drag Force

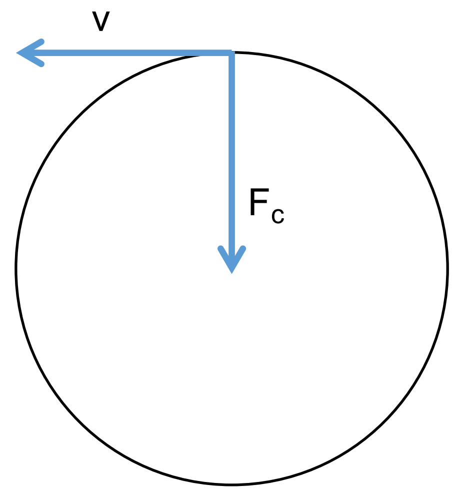
# Centripetal Motion

$$\omega = \frac{2\pi}{T}$$

$$v = \frac{2\pi r}{T} = \omega r$$

$$a = \frac{v^2}{r} = r\omega^2$$

**ω, angular velocity**
**T, period of rotation**
**r, radius**
**α, angular acceleration**



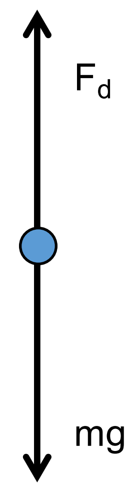

$$\alpha = \frac{d\omega}{dt} = 0 \text{ when } v \perp a$$

# Drag Force

$$F_d = k\,m\,v$$

$$F_d = k\,m\,v^2$$

$$v_t = \frac{g}{k}\ OR\ vt = \sqrt{g/k}$$




$$F_d = \frac{1}{2} c_d\, A\, \rho\, v^2$$

$c_d$, drag coefficient
ρ, density of medium
A, cross-sectional area of object
v, speed

# Orbital Motion

$$F = G\frac{m_1\, m_2}{r^2}$$

G=6.67408*10$^{-11}$ m^3 kg^-1 s^-2

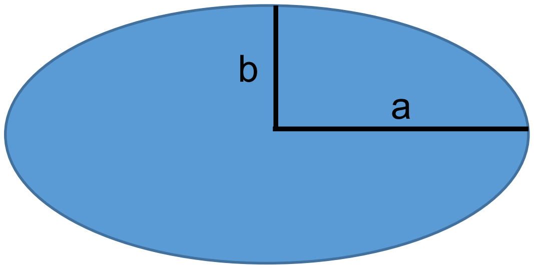
$$U(r) = -G\frac{m_1\, m_2}{r}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2}$$

$$v_{escape} = \sqrt{(2GM)/R}$$

# Kepler's 1st Law

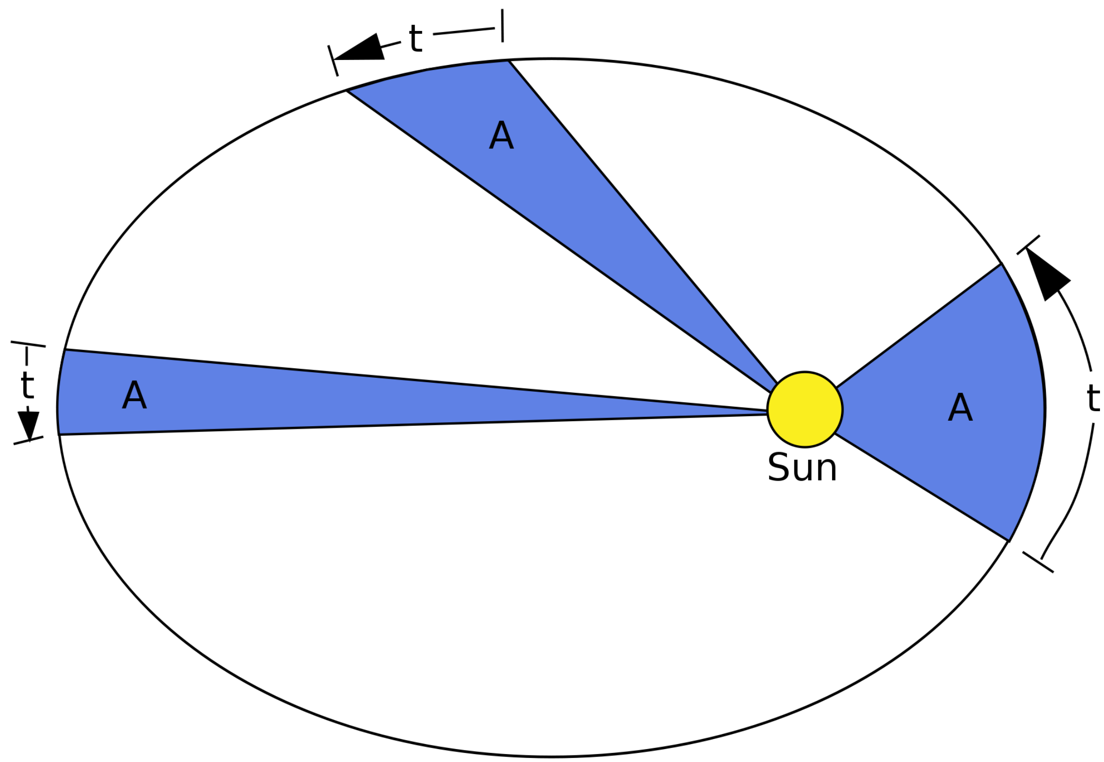
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$




$$r_{min} = a(1 - \varepsilon)$$

$$r_{max} = a(1 + \varepsilon)$$

# Kepler's 2$^{nd}$ Law



# Kepler's 3$^{rd}$ Law

$$\boxed{T^2 = \left(\frac{4\pi^2}{Gm}\right) a^3}$$

# Lagrangian and Hamiltonian Mechanics

# Lagrangian Equations of Motion: How to

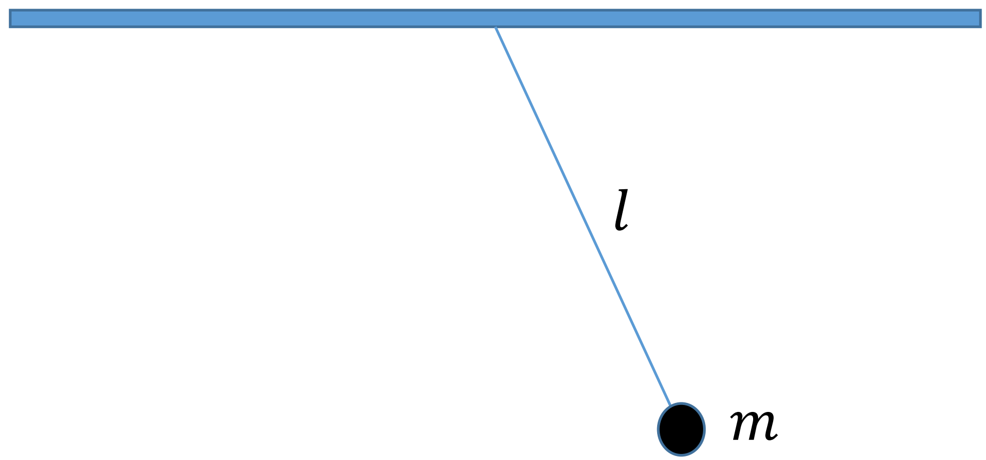
1. Draw a picture
2. Determine the number of degrees of freedom and define variable names for the degrees of freedom (Generalized Coordinates)
3. Write $x$, $y$, and $z$ for each particle in terms of the generalized coordinates
4. Evaluate $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of generalized coordinates and velocities
5. Write Lagrangian: $L = \mathrm{T} - \mathrm{U}$
6. Solve LEM

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

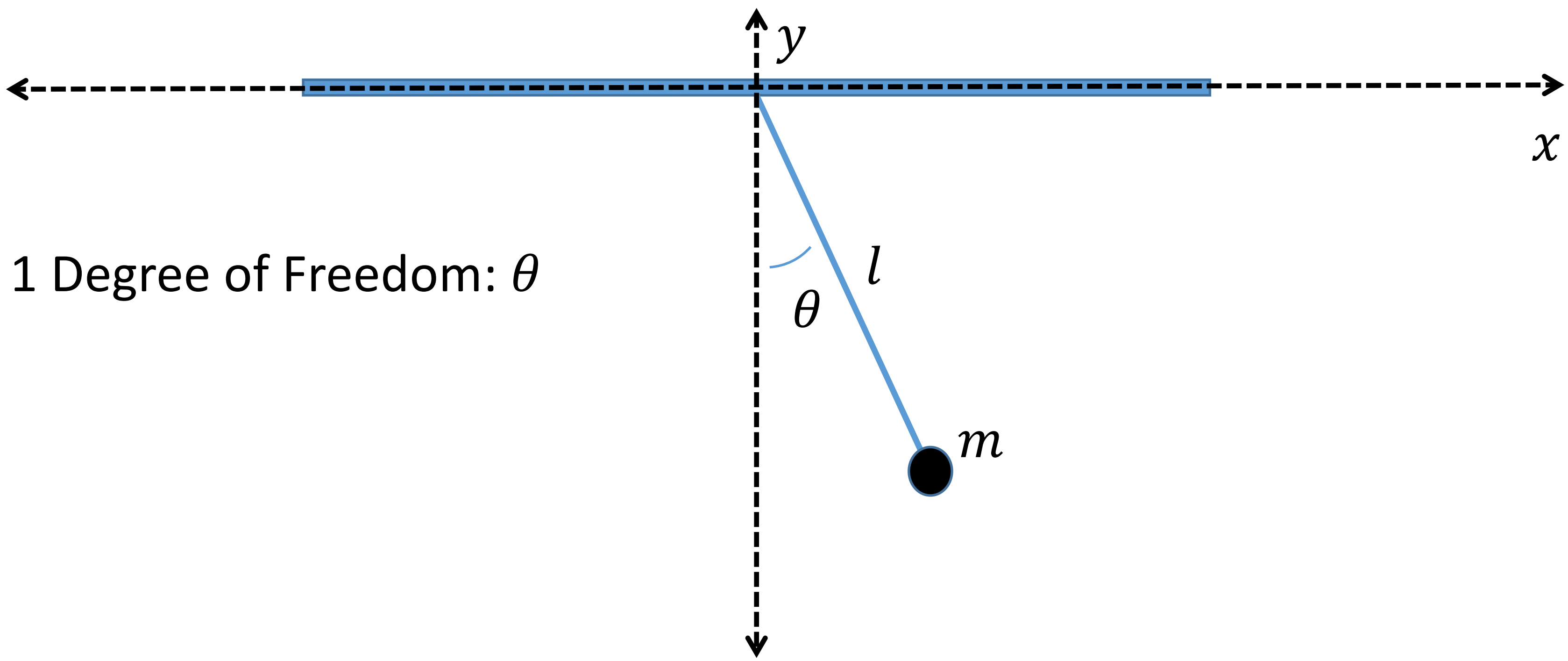
# Example Problem

A ball of mass $m$ hangs from a rod of length $l$ and swings back and forth in the xy-plane. Find the Lagrangian Equations of Motion.

# 1. Draw a picture

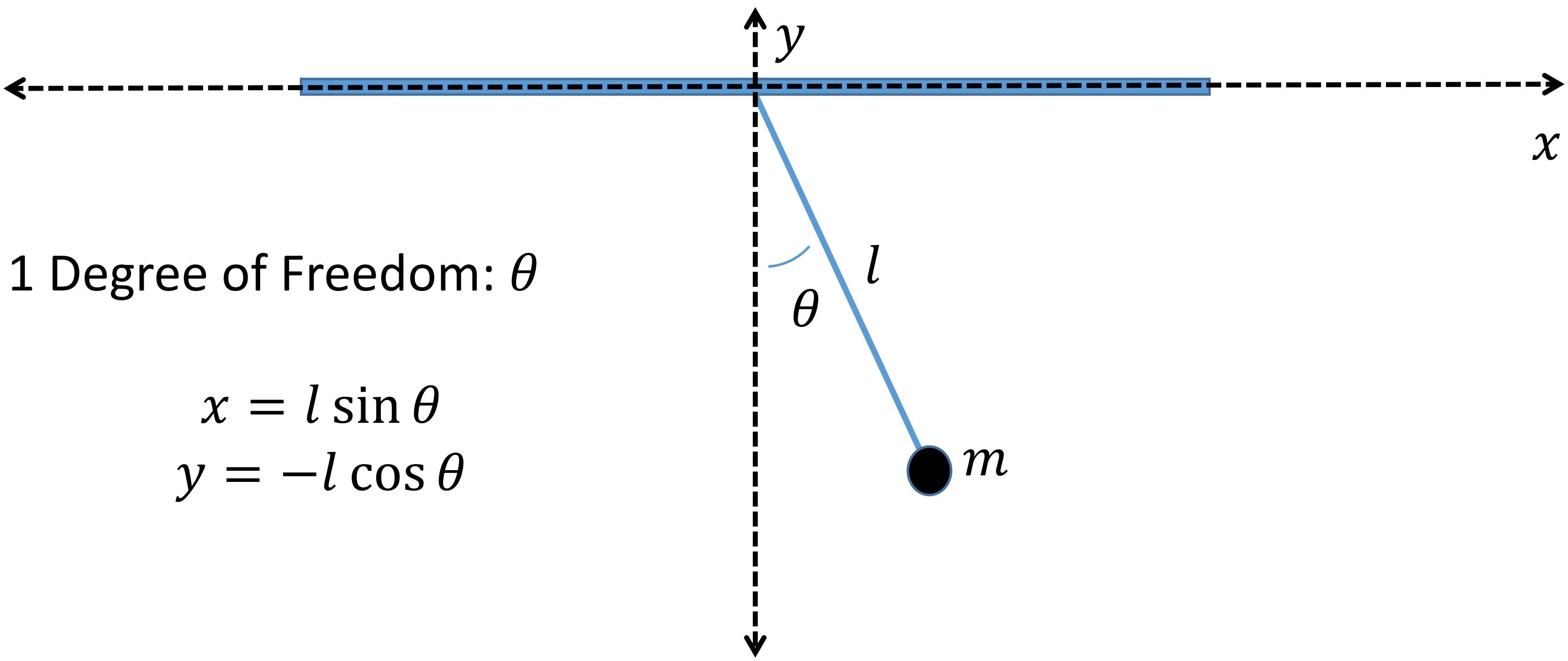


2. Determine the number of degrees of freedom and define variable names for the degrees of freedom (Generalized Coordinates)




1 Degree of Freedom: $\theta$

# 3. Write $x$, $y$, and $z$ for each particle in terms of the generalized coordinates




1 Degree of Freedom: $\theta$

$$x = l \sin\theta$$
$$y = -l \cos\theta$$

## 4. Evaluate $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of generalized coordinates and velocities

Recall:

$$x = l \sin\theta$$
$$y = -l \cos\theta$$

Therefore:

$$\dot{x} = l \cos\theta\,\dot{\theta}$$
$$\dot{y} = l \sin\theta\,\dot{\theta}$$

# 5. Write Lagrangian: $L = \mathrm{T} - \mathrm{U}$

Kinetic Energy ($T$)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m\left((l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right)$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential Energy ($U$)

$$U = mgy$$

$$U = -mgl\cos\theta$$

Lagrangian ($L$)

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

## 6. Solve LEM: $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$

Recall: $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Therefore:

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = ml^2\ddot{\theta}$$

# Solution

We note

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

# Hamiltonian Equations of Motion: How To

1. Find Lagrangian
2. Find Hamiltonian
    1. Find Generalized Momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$
    2. Hamiltonian: $H = \sum p_i \dot{q}_i - L$
    3. Make H "dotless"
3. Solve HEM

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

# 1. Find Lagrangian

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta$$

# 2.1 Find Generalized Momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

Note: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Therefore,

$$p_\theta = ml^2\dot{\theta}$$

## 2.2 Hamiltonian: $H = \sum p_i \dot{q}_i - L$

Recall:

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta \text{ and } p_\theta = ml^2 \dot{\theta}$$

Therefore,

$$H = ml^2 \dot{\theta}\dot{\theta} - \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos\theta$$

$$H = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos\theta$$

# 2.3 Make H "dotless"

Recall:

$$H = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

Making $H$ "dotless": $p_\theta = ml^2\dot{\theta}$ -> $\dot{\theta} = \frac{p_\theta}{ml^2}$

$$H = \frac{1}{2} m l^2 (\frac{p_\theta}{ml^2})^2 - mgl\cos\theta$$

$$H = \frac{1}{2}\frac{{p_\theta}^2}{ml^2} - mgl\cos\theta$$

# 3. Solve HEM

Hamiltonian: $H = \frac{1}{2}\frac{{p_\theta}^2}{ml^2} - mgl\cos\theta$

Solution: $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Therefore,

$$\dot{\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{p}_\theta = -mgl\sin\theta$$

# Note

Note: $p_\theta = lmv = l^2 m\dot{\theta}$ -> $\dot{p}_\theta = l^2 m\ddot{\theta}$

$$l^2 m\ddot{\theta} = -mgl\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

(Same as Lagrangian)

# Other Notes

*Iff the Lagrangian is independent of a coordinate q, the corresponding conjugate momentum* $\frac{\partial L}{\partial \dot{q}}$ *is conserved. (Time derivative is zero)*

*Iff the Hamiltonian is independent of a coordinate q, the corresponding conjugate momentum p is conserved.*