Chapter 13: Partial Differential Equations (PDE’s)

First of all, this section is very, very difficult. And new to you. But it’s also super cool.

PDE’s → there is more than one independent variable.

Example: \( \nabla^2 \phi = 0 \) → \( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \) has independent variables \( x, y, \) and \( z \).

The dependent variable is \( \phi \).

As with ODE’s, the general procedure is to hope that somebody else can tell you the answer before you even start the problem. But, sometimes you still have to do it yourself.

In physics, there are a zillion relevant PDE’s. Example include heat transfer, the equations that describe the motion of waves (e.g., sound, or light, which is a wave of electric field), Schrodinger’s equation, etc.

Most of these equations include a \( \nabla^2 \) somewhere, which always results in derivatives with respect to \( x, y, \) and \( z \). In addition, if you want to know the temperature (or whatever), you not only have to specify where you want to know \( \phi \) (i.e., at \( x, y, \) and \( z \)), but often you also have to specify when you wanted to know \( \phi \) (i.e., \( t \)). So, a lot of PDE’s have 4 independent variables. Naturally, there are even more kinds of problems than just finding temperature as a function of these 4 variables. You might instead have an equation that, if solved, could tell you pressure as a function of temperature and density.

**Separation of Variables**

Whenever possible, we solve PDE’s by a method called “separation of variables”, which is unfortunately not anything like the “separation of variables” we used to solve ODE’s. For PDE’s “separation of variables” is a nickname for a method actually called “Eigenfunction decomposition”.

**The Heat Equation**

Let’s start with a simplified form of “the heat equation”. This equation is about conduction: how the temperature in one part of an object affects the temperature in other parts. The basic equation is: \( k \nabla^2 T = \frac{\partial T}{\partial t} \). This equation assumes that there are no sources of energy embedded in the object, so it is already somewhat simplified. We will further simplify it by saying that our object is only 2D, and also by saying that the temperature profile is “steady” (i.e., that it is not a function of time). With these simplifications, the equation becomes merely \( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \).

Notice that the conductivity of the material, \( k \), cancelled out altogether as a result of making it
“steady”. As you imagine, this equation is not that interesting yet. In fact, it has an infinite number of answers.

In order to even get started, we have to recognize that there must be some cause for the temperature to actually be one way instead of another. For most problems, that means that we need to specify boundary conditions. Looking at this equation, I see two derivatives with respect to \( x \), and two with respect to \( y \). That means that I need to specify a total of four boundary conditions before this even becomes a real physics problem that I might want to bother with. So, here is the “real” problem I want to solve:

My object is a rectangular sheet of metal (the kind of metal apparently doesn’t matter, since \( k \) is gone). I will hold the bottom edge against something hot, and the other three edges against something cold. In other words, I will completely specify the temperatures around the edges. As you might imagine, with these boundaries, the temperature of parts of the plate near the bottom edge will likely be hotter than those near the other edges. That’s our goal: to find \( T(x, y) \).

Let’s write out the boundary conditions explicitly:

\[
T(x = 0) = T_c \\
T(x = L_1) = T_c \\
T(y = 0) = T_h \\
T(y = L_2) = T_c
\]

It is customary, but not strictly necessary, to algebraically manipulate the equation and all the boundary conditions in such a way that the result has no units. This is done so that having created the solution in a generic way, we can apply it to multiple similar situations. It is analogous to solving Analyt I problems symbolically instead of numerically.

Here is the usual way of removing units from equations. We create new variables that are linearly related to our original independent and dependent variables:

\[
x^* = \frac{x}{L_1} \quad \rightarrow \quad x = x^* L_1 \\
y^* = \frac{y}{L_1} \quad \rightarrow \quad y = y^* L_1 \\
N = \frac{L_2}{L_1} \quad \rightarrow \quad L_2 = NL_1 \\
\Theta = \frac{T - T_c}{T_h - T_c} \quad \rightarrow \quad T = (T_h - T_c) \Theta + T_c
\]
Notice that none of the four new variables has any units. Simple substitution directly into our basic equation and also into our boundary conditions results in:

\[
\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = 0
\]

\[
\Theta(x^* = 0) = 0
\]

\[
\Theta(x^* = 1) = 0
\]

\[
\Theta(y^* = 0) = 1
\]

\[
\Theta(y^* = N) = 0
\]

In other words, we will really solve for \(\Theta(x^*, y^*)\). Once done, we can convert it back into \(T\), if we care, using the last equation on the previous page.

At this point, it also customary to admit that we are too lazy to bother writing out all the “stars”. So, even though it’s confusing, we’ll write \(x\) when we really mean \(x^*\). Now, we can start the real work. First, “Separation of Variables”: this means that we cross our fingers and hope that the answer is the product of two separate answers, each of which is itself a function of only one of the independent variables. Let’s call these two functions \(X(x)\) and \(Y(y)\).

So, we’re hoping that \(\Theta = X(x)Y(y)\).

Let’s substitute this bit of this hopefulness back into the main equation:

\[
\frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial^2 X(x)Y(y)}{\partial x^2} = Y(y) \frac{\partial^2 X(x)}{\partial x^2},
\]

\[
\frac{\partial^2 \Theta}{\partial y^2} = \frac{\partial^2 X(x)Y(y)}{\partial y^2} = X(x) \frac{\partial^2 Y(y)}{\partial y^2}
\]

\[
\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = Y(y) \frac{\partial^2 X(x)}{\partial x^2} + X(x) \frac{\partial^2 Y(y)}{\partial y^2} = 0
\]

So,

\[
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} = 0
\]

You might see why they call this “separation of variables”… all the \(x\)’s are together in one group (both the variable \(x\) and the function \(X\)), and all the \(y\)’s are together, too.

If some function of only \(x\) plus another function of only \(y\) add up to zero, always, then logically it must be true that each function is a constant, and one is the negative of the other. I’ll write it this way:
\[
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = -k^2 \quad \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} = +k^2
\]

As you can see, I called the constant \(k^2\), but it has nothing to do with the conductivity \(k\) that we saw earlier.

In other words, if my hopes work out, then what I really need to do is solve these two separate ordinary differential equations, then multiply the two results together. The decision to call the constant \(-k^2\) instead of something more inspirational such as \(C_1\) comes from hindsight, from having done the problem already and noticing that if you called it \(C_1\), then the answer has a \(\sqrt{-C_1}\) in it. So, by calling it \(-k^2\), we’re hoping to make the answer look simpler in the long run.

Let’s now solve these two equations separately:

\[
\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k^2 \quad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = +k^2
\]
\[
\frac{\partial^2 X(x)}{\partial x^2} = -k^2 X \quad \frac{\partial^2 Y}{\partial y^2} = +k^2 Y
\]
\[
X'' + k^2 X = 0 \quad Y'' - k^2 Y = 0
\]

From chapter 8, we use the auxiliary equation "D" method to discover:

\[
X = A \cos(kx) + B \sin(kx) \quad Y = Ce^{ky} + De^{-ky}
\]

\[
\Theta = (A \cos(kx) + B \sin(kx)) \left( Ce^{ky} + De^{-ky} \right)
\]

We now have 5 unknowns to solve for \((A, B, C, D, k)\), using our 4 boundary conditions. I hope one of the unknowns magically disappears!

**BC #1 (left edge):** \(\Theta(x = 0) = 0 \rightarrow \text{therefore } A = 0 \rightarrow \Theta = (B \sin(kx)) \left( Ce^{ky} + De^{-ky} \right)\)

**BC #2 (right edge):** \(\Theta(x = 1) = 0 \rightarrow \text{0 = } (B \sin(k)) \left( Ce^{ky} + De^{-ky} \right)\). The only way for this to happen is if \(B\) is always zero, or if \(\sin(k)\) is always zero. If \(B = 0\), then we have no solution left whatsoever, so let’s examine \(\sin(k) = 0\). In this case, it must then be true that \(k\) is some integer multiple of \(\pi\). In fact, it can be any and every integer multiple of \(\pi\): \(k = n\pi\)

So, our solution so far is \(\Theta = \sum_{n=1}^{\infty} (B_n \sin(n\pi x)) \left( Ce^{n\pi y} + De^{-n\pi y} \right)\)

**BC #3 (top edge):** \(\Theta(y = N) = 0 \rightarrow \text{0 = } \left( Ce^{kn} + De^{-kn} \right)\), or \(D = -\frac{Ce^{N+kn} - Ce^{-N}}{e^{-n\pi N}}\).
So, our total answer so far is:

\[
\Theta = \sum_{n=1}^{\infty} \left( B_n \sin[n \pi x] \right) \left( Ce^{n \pi y} - C e^{n \pi y} e^{-n \pi y} \right)
\]

\[
\Theta = \sum_{n=1}^{\infty} \left( B_n \sin[n \pi x] \right) \left( Ce^{-n \pi y} e^{n \pi y} - Ce^{n \pi y} e^{-n \pi y} \right)
\]

If we review chapter two a little bit, we might recognize that this can be simplified. Specifically,

\[
\sinh(x) = \frac{e^x - e^{-x}}{2}.
\]

Plugging this in results in this “simplified” result so far:

\[
\Theta = \sum_{n=1}^{\infty} \left( B_n \sin[n \pi x] \right) \left( E_n \sin[n \pi (N - y)] \right), \text{ where } E_n = -2Ce^{n \pi y}.
\]

To save myself some trouble, I’ll call the product \( B_nE_n \) to be some new function \( F_n \) (this is what eliminates my “5th” unknown, by the way):

\[
\Theta = \sum_{n=1}^{\infty} \left( F_n \sin(n \pi x) \right) \sinh\left(n \pi (N - y)\right)
\]

I have one more BC: BC #4 (bottom edge): \( \Theta(y = 0) = 1 \)

\[
1 = \sum_{n=1}^{\infty} \left( F_n \sin(n \pi x) \right) \sinh\left(n \pi (N - 0)\right)
\]

This is a lot like the Fourier series problems we’ve done recently. Again, to save myself some cramping in my hand, I’ll once again invent a new letter: \( b_N = F_n \sinh(n \pi N) \).

In other words, I get:

\[
1 = \sum_{n=1}^{\infty} \left( b_n \sin(n \pi x) \right)
\]

If this were a Fourier series problem it would have a \( 1/L \) in front of it. So, this thing on the right hand side is the Fourier series for \( f(x) = 1 \) using \( L = 1 \). To complete the final steps of this problem, we need to think of this Fourier series as representing a repeating function. Since it only has sine terms in it, it better be an odd repeating function, as opposed to an even one:
Solving this for $b_n$ as we did for all the other Fourier series last week:

$$b_n = \frac{1}{L} \int_{-1}^{0} (-1) \cdot \sin \left( \frac{n\pi x}{L} \right) dx + \frac{1}{L} \int_{0}^{1} (+1) \cdot \sin \left( \frac{n\pi x}{L} \right) dx = \frac{2}{n\pi} \left(1 - \cos(n\pi) \right)$$

The function $\cos(n\pi)$ is interesting, and might be simplified if we’re lucky:

If $n$ is an even number, then $\cos(n\pi) = 1$, and so $b_n = 0$.
If $n$ is an odd number, then $\cos(n\pi) = -1$, and $b_n = \frac{4}{n\pi}$.

I’ll take this $b_n$, and plug it back into $b_n = F_n \sinh(n\pi N)$ to find $F_n$, and

So, we have arrived at some version of our complete answer:

$$\Theta = \sum_{n=\text{odd}}^{\infty} \left( \frac{4}{n\pi} \frac{\sinh \left( n\pi (N-y) \right)}{\sinh \left( n\pi N \right)} \sin(n\pi x) \right).$$

Our last step would be to convert $n$ into some other form $m$ that isolates the odd values for us, as we did last week. In this case, odd values of $n$ are generated steps of $2m - 1$. So, for example, $m = 1 \rightarrow n = 1, \quad m = 2 \rightarrow n = 3, \quad m = 3 \rightarrow n = 5$, etc.

$$\Theta = \sum_{m=3}^{\infty} \left( \frac{4}{(2m-1)\pi} \frac{\sinh(2m-1)\pi(N-y))}{\sinh(2m-1)\pi N} \sin(2m-1)\pi x) \right)$$

That’s been a lot of work here. It requires us to combine parts from virtually every separate topic that we’ve studied so far this semester. We used series, complex numbers, chain rules, integrals, differential equations, and Fourier series. The only thing we didn’t use was matrices.

See associated Mathematica sheet for generation of this 2D (contour) plot of the solution: